The Chernoff Bound

$X_1, X_2, .. X_n$ - independent Bernoulli trials s.t.

$Pr(X_i = 1) = p \quad Pr(X_i = 0) = 1 - p$

$X = \sum_i X_i \rightarrow X$ is binomial distribution

$Pr(X_i = 1) = p_i$

↳ Poisson trials

Chance to not achieve some performance is small to guarantee this we study $Pr[X > (1+\delta)\mu]$, where $E(X) = \mu$.

$E(e^{tX})$ - moment generating function of X.

Chernoff bound idea — 1) take moment generating function
2) Apply Markov inequality to it.

Sum of independent random variables appears in the exponent, this turns into the product of random variables whose expectation we then bound.

Theorem $X_1, .. X_n$ - independent Poisson trials, $Pr[X_i = 1] = p_i$

$$Pr[X > (1+\delta)\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$$

Proof
1) Write moment generating function
2) Apply Markov inequality.
3) Expectation of Product is product of expectation
4) Pick desired value of t.

$$\Pr[X > (1+\delta)\mu] = \Pr[\exp(tX) > \exp(t(1+\delta)\mu)]$$

$$\Pr[X > (1+\delta)\mu] < \frac{E(\exp(tX))}{\exp(t(1+\delta)\mu)}$$

strict because $p_i \neq 0,1$.

$$E(\exp(tX)) = E(\exp(t\sum_i X_i)) = E(\prod \exp(tX_i))$$

$$\Pr[X > (1+\delta)\mu] < \frac{E(\prod(\exp(tX_i)))}{\exp(t(1+\delta)\mu)} = \frac{\prod(E\exp(tX_i))}{\exp(t(1+\delta)\mu)}$$

$$= \frac{\prod(p_i e^t + 1 - p_i)}{\exp(t(1+\delta)\mu)}$$

$1 + x < e^x \qquad x = p_i(e^t - 1)$

$$\Pr[X > (1+\delta)\mu] < \frac{\prod \exp(p_i(e^t-1))}{\exp(t(1+\delta)\mu)} = \frac{\exp(\sum_i p_i(e^t-1))}{e^{t(1+\delta)\mu}} =$$

$$= \frac{\exp((e^t-1)\mu)}{\exp(t(1+\delta)\mu)}$$

$t = \ln(1+\delta)$

$$\Pr[X > (1+\delta)\mu] = \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu .$$

$\boxtimes$

<u>Def</u> $\quad F^+(\mu,\delta) = \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu$

Example Arkansas Aardvarks.

Выигрывают с вер $1/3$, вер. выиграть больше половины

$$Pr\left(Y_n > \frac{n}{2}\right) < F^+\left(\frac{n}{3}, \frac{1}{2}\right) = (0.965)^n .$$

Theorem

$$Pr\left[X < (1-\delta)\mu\right] < \exp\left(\frac{-\mu\delta^2}{2}\right)$$

Proof. Proof is similar to previous proof.

$$Pr\left[X < (1-\delta)\mu\right] = Pr\left[-X > -(1-\delta)\mu\right] = Pr\left[e^{-tX} > e^{-t(1-\delta)\mu}\right] <$$

$$< \frac{E(e^{-tX})}{e^{-t(1-\delta)\mu}} = \frac{E\left(\prod e^{-tX_i}\right)}{e^{-t(1-\delta)\mu}} = \frac{\prod\left(E e^{-tX_i}\right)}{e^{-t(1-\delta)\mu}} =$$

$$= \frac{\prod\left(p_i e^{-t} + (1-p_i)\right)}{e^{-t(1-\delta)\mu}} \leq \frac{\prod\left(e^{p_i(-1+e^{-t})}\right)}{e^{-t(1-\delta)\mu}} = \frac{e^{(e^{-t}-1)\mu}}{e^{-t(1-\delta)\mu}}$$

Let $t = \ln\left(\frac{1}{1-\delta}\right)$

$$Pr\left[X < (1-\delta)\mu\right) < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$$

$$(1-\delta)^{1-\delta} > \exp\left(-\delta + \delta^2/2\right)$$

$$\Rightarrow Pr\left[X < (1-\delta)\mu)\right] < e^{-\frac{\delta^2}{2}\mu}.$$

☒

Def $F^-(\mu, \delta) = \exp\left(-\frac{\mu\delta^2}{2}\right)$.

Note: ① apply Chernoff technique to $-X$, not to $n-X$.

② $\ln(1-\delta)$ has good expansion while $\ln(1+\delta)$ not.

~~Box~~ Example Arkansas Aadvarks again

$$\Pr[Y_n < \tfrac{n}{2}] < F^-(0.75n, \tfrac{1}{3}) < (0.9592)^n$$

Предыдущие теоремы не зависят от n, только от μ и δ. $\overset{\text{trials}}{1000}$, $p_i = 0.02$ = $\overset{\text{trials}}{100}$ $p_i = 0.2$

Def $\Delta^+(\mu,\varepsilon)$ is such value that

$$F^+(\mu, \Delta^+(\mu,\varepsilon)) = \varepsilon$$

Similarly $\Delta^-(\mu,\varepsilon)$: $F^-(\mu, \Delta^-(\mu,\varepsilon)) = \varepsilon$

$$\Delta^-(\mu,\varepsilon) = \sqrt{\frac{2\ln(1/\varepsilon)}{\mu}}.$$

$$e^{-\frac{\mu\delta^2}{2}} \le \varepsilon \Rightarrow \frac{\mu\delta^2}{2} = \ln\frac{1}{\varepsilon} \Rightarrow \delta = \sqrt{\frac{2\ln\frac{1}{\varepsilon}}{\mu}}$$

Theorem For $\delta \in [0, U]$

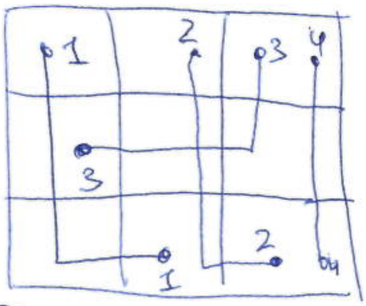
$$F^+(\mu,\delta) \le \exp(-c(U)\mu\delta^2), \text{ where}$$

$c(U) = [(1+U)\ln(1+U) - U]/U^2.$

For $U = 2e-1$ $F^+(\mu,\delta) < \exp\left(-\frac{\mu\delta^2}{4}\right)$

Given $\delta < 2e-1 \Rightarrow \Delta^+(\mu,\varepsilon) < \sqrt{\frac{4\ln\frac{1}{\varepsilon}}{\mu}}.$

Wiring problem



Plan

1) Write ILP
2) Relax to LP, and solve LP
3) Round obtained solution.

ILP

min ω

$x_{i0}, x_{i1} \in \{0,1\}$

$x_{i0} + x_{i1} = 1$

$$\sum_{i \in T_{b0}} x_{i0} + \sum_{i \in T_{b1}} x_{i1} \le \omega$$

LP

$\Rightarrow$ $0 \le x_{i0}, x_{i1} \le 1$

$Pr[\bar{x}_{i0} = 1] = x_{i0}$ $\quad Pr[\bar{x}_{i1} = 1] = x_{i1}$

Theorem $\varepsilon \in \mathbb{R}$, с вероятностью $1-\varepsilon$ решение для wiring problem полученное с помощью вероятностного округления удовлетворяет:

$$\omega_S \le \hat{\omega}\left(1 + \Delta^+\left(\hat{\omega}, \frac{\varepsilon}{2n}\right)\right) \le \omega_0\left(1 + \Delta^+\left(\omega_0, \frac{\varepsilon}{2n}\right)\right).$$

Док-во: покажем, что вероятность для конкретной стенки, что $\omega_S(b) > \hat{\omega}\left(1+\Delta^+\left(\hat{\omega}, \frac{\varepsilon}{2n}\right)\right)$ не больше $\frac{\varepsilon}{2n}$.

Мы знаем, что

$$\sum_{i \in T_{b0}} \hat{x}_{i0} + \sum_{i \in T_{b1}} \hat{x}_{i1} \le \hat{\omega}$$

$$E(\omega_{S_b}) = \sum E(\bar{x}_{i0}) + \sum E(x_{i1}) = \sum \hat{x}_{i0} + \sum \hat{x}_{i1} \le \hat{\omega}$$

Далее все следует из определения $\Delta^+(\hat{\omega}, \frac{\varepsilon}{2n})$.

$$P_2 [w_S(b) > \hat{\omega}(1 + \Delta^+(\hat{\omega}, \frac{\varepsilon}{2n}))] \le \frac{\varepsilon}{2n}$$

~~Далее по~~ Вер. объединения не больше суммы вероятностей. $\boxtimes$

Congestion Minimization