Lower bounds(HW).

1. Given a graph G and an integer k, the Independent Dominating Set problem asks for a set of exactly k vertices that is both an independent set and a dominating set. Prove that Independent Dominating Set is $W[2]$-complete.

2. In this exercise we will work out an alternative, purely combinatorial construction of k-paradoxical tournaments(турнир без доминирующего множества размера k) of size $2^{poly(k)}$.

   - Find a 2-paradoxical tournament T^* on 7 vertices.
   - Assume we are given some tournament T. Construct a tournament T' as follows. The vertices of T' are triples of vertices of T, i.e., $V(T') = V(T) \times V(T) \times V(T)$. Let us consider a pair of triples $u_1 = (a_1, b_1, c_1)$ and $u_2 = (a_2, b_2, c_2)$, where $a_1 \neq a2$, $b_1 \neq b_2$, and $c_1 \neq c_2$. Consider now pairs $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$, and count for how many of them the edge in T was directed from the vertex with subscript 1 to the vertex of subscript 2 (e.g., from a_1 to a_2). If for at least 2 pairs this was the case, we put $(u_1, u_2) \in E(T')$, and otherwise we put $(u_2, u_1) \in E(T')$. For all the pairs of triples where at least one of the coordinates is the same, we put the edge between the triples arbitrarily. Prove that if T was k-paradoxical, then T' is $\lfloor \frac{3k}{2} \rfloor$-paradoxical.
   - Define the sequence $T_0, T_1, T_2 \ldots$ as follows: $T_0 = T*$, and for $m \geq 1$ the tournament T_m is constructed from T_{m-1} using the construction from the previous point. Prove that $|V(T_m)| = 7^{3^m}$ and that T_m is $g(m)$-paradoxical for a function $g(m) \in \Omega((\frac{3}{2})^m)$.
   - Using the previous point, provide an algorithm that, given an integer k, constructs a k-paradoxical tournament of size $2^{O(k^{\log_{3/2} 3})} \leq 2^{O(k^{2.71})}$. The construction should work in time polynomial with respect to the size of the constructed tournament.

3. Given a bipartite graph G with bipartite classes $A, B \subseteq V(G)$ and an integer k, the Hall Set problem asks for a Hall set of size at most k, that is, a set $S \subseteq A$ of size at most k such that $|N(S)| < |S|$. Show that Hall Set is W[1]-hard.