On list inicdentor (k, l)-coloring

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Annotation

A proper incidentor coloring is called (k, l)-coloring if the difference between the colors of final and initial incidentors is at least k and at most l. In the list variant the additional restriction is added: the color of each incidentor must belong to a set of feasible colors of the arc. Such variant has sense if the set of feasible colors for each arc is an integer interval. The minimum length of the interval that guarantees the existence of list incidentor (k, l)-coloing is called a list incidentor (k, l)-chromatic number. In this work we prove bounds on the list (k, l)-chromatic number for multigraphs of degree 2 and 4.

Introduction

Let us call a finite subset of natural consecutive numbers an *interval*. A *length* of an interval $[a, b]$ for integer $a \le b$ equals to $b - a + 1$. In a directed graph $G = (V, E)$ a pair (u, e) consisting of a vertex u and an arc e incident to this vertex is called an *incidentor*; it is convenient to consider the incidentor as the half of the arc e incident with the vertex u. So an arc $e = uv$ contains two incidentors (u, e) and (v, e) called *initial* and *final* respectively. Such incidentors are *mated* to each other. Two incidentors adjoining the same vertex we will call *adjacent*. Given an incidentor i we denote by i' the incidentor mated to i.

A *coloring* of some incidentor set is a mapping from this set to the set of colors (integer positive numbers). An incidentor coloring is called *proper* if all adjacent incidentors are colored in different colors.

An incidentor coloring is a (k, l)-*coloring* if it is proper and the difference between the colors of final and initial incidentors is at least k and at most

l. A minimum number of colors necessary for incidentor (k,l)-coloring of a given graph is called its incidentor (k,l)-chromatic number.

A coloring is called a *list coloring* if for any arc a set of admissible colors is given and all inicidentors have to be colored only by admissible colors. In general case, a list chromatic number is a minimum cardinality of a set of admissible colors of each arc such that the list coloring exists. But in case of incidentor (k,l)-coloring (where $k > 0$) such definition is senseless, because even the infinite list of colors $\{1, l+2, 2l+3, \ldots\}$ doesn't allow the list (k,l)-coloring. In order to exclude such situation we need an additional restriction on the color lists. It looks natural to demand that admissible colors for every arc form an integer interval. So a *list incidentor* (k,l)-chromatic number is a minimum t such that for any assignment of intervals of admissible colors of length at least t for any arc there exists a (k,l)-coloring of incidentors using only admissible colors. We denote this number by $\chi_{k,l}^{list}(G)$.

Often it is convenient to consider an incidentor coloring as a coloring of arcs by pairs of colors; namely, we write $f(e) = (a,b)$, if the initial incidentor of the arc e is colored by the color a, and the final one — by the color b.

The problem of incidentor coloring was first formulated in [7] as a helpful model for the problem of optimisation of transfering messages in a local communication network. In [5] a conception of the (k,l)-coloring was introduced, which was studied further in [2, 3, 4, 8, 9, 10, 11]. A list k-coloring (that is equivalent to (k,∞)-coloring in our terminology) was first studied by V.G.Vizing in [1]. It was proved there that for a graph of degree Δ exactly $\Delta + k$ colors are sufficient, if Δ is even and at most $\Delta + k + 1$, if Δ is odd. Also the following conjecture was posed: for any graph $\Delta + k$ colors are enough. In case $\Delta = 3$ this conjecture was proved in [12]. The problem of list incidentor (k,l)-coloring has never been studied before.

In this work we first prove simple general bounds on $\chi_{k,l}^{list}(G)$ and consider case of multigraphs of degree 2. The main result of the work is the following conjecture and the proof of this conjecture for multigraphs of degree 4:

Conjecture. *For any multigraph of degree Δ and $l \leq k + \Delta - 1$ holds:*

$$\chi_{k,l}^{list}(G) \leq 2k + 2\Delta - l - 1.$$

1 Preliminary results

First we prove some general bounds on $\chi_{k,l}^{list}(G)$.

Theorem 1. *For any multigraph of degree Δ holds:*

$$\chi_{k,k}^{list}(G) \leq k + 2\Delta - 1.$$

Proof. Note, that if an arc e has a list $[a, a + k + 2\Delta - 2]$ than its incidendtors can be colored in $2\Delta - 1$ ways like $f(e) = (a + i, a + i + k)$, where $i = 0, \ldots, 2\Delta - 2$. Each incidentor of an arc is adjacent to at most $\Delta - 1$ other incidentors, so at least one of $2\Delta - 1$ ways of coloring remains proper. Since this is correct for any arc the proper coloring can be easily constructed using "greedy" algorithm. Theorem 1 is proved.

Also we can show the following lower bound on $\chi_{k,l}^{list}(G)$:

Proposition 1. *For any multigraph of degree Δ holds:*

$$\chi_{k,l}^{list}(G) \geq \Delta + k.$$

Proof. A graph in which all arcs have the same set of admissible colors and which has at least one vertex with only outcoming arcs can be easily used to show that intervals of length $\Delta + k - 1$ is not enough. Hence, $\chi_{k,l}^{list}(G) \geq \Delta + k$. Proposition 1 is proved.

In case $\Delta = 2$ the bound from Theorem 1 can be improved (when $l > 0$). Notice that incidentor $(0, 0)$-coloring coincides with the edge coloring of a graph and is not studied in this work.

Theorem 2. *For any $l > 0$ and $k \leq l$ list incidentor (k, l)-chromatic number equals to $k + 2$ for multigraphs of degree 2.*

Proof. First we consider the case $k > 0$. Then we may assume that $l = k$. For every arc we have a set of admissible colors of type $[a, a+k+1]$ for some a. As in the previous theorem, there are two ways to color incidentors of such arc e: either $f(e) = (a, a + k)$, or $f(e) = (a + 1, a + k + 1)$. Call the colors a and $a + 1$ *essential* for the initial incidentor of the arc e and colors $a + k$ and $a + k + 1$ for the final incidentor of the arc e.

Consider a component of the graph G. If it is a path then the coloring can be easily consructed using "greedy" algorithm along the path since on each step the uncolored arc is adjacent to at most one colored incidentor.

Now let the component be a cycle. Assume there is a vertex such that sets of admissible colors of incidentors adjacent to this vertex are different.

Then the coloring is constructed in the same way as for paths: we just need to pick the coloring of the first arc in such a way that both essential colors of the incidentor adjacent to this arc remain free. Assume now this is not the case and for each vertex v essential colors of both adjacent to v incidentors coincide. Denote by $a(v)$ the least of these colors. Choose the direction of the cycle $v_1, \ldots, v_n$ in an arbitrary way. Note, that if i-th arc is $v_i v_{i+1}$ (i.e. its direction coincides with the direction of the cycle) then $a(v_{i+1}) = a(v_i) + k$; otherwise, $a(v_{i+1}) = a(v_i) - k$. Since $v_{n+1} = v_1$ and $k \geq 1$, the number of arcs codirectional with the cycle equals to the number of arcs oppositely directed. Therefore the cycle is even and $n = 2t$. For all $j = 1, \ldots, t$ we color incidentors of $2j$-th arc by the least of essential colors of its incidentors (i.e. by colors $a(v_{2j})$ and $a(v_{2j+1})$), and incidentors of $(2j-1)$-th arc — by the greatest of essential colors of its incidentors (i.e. by colors $a(v_{2j-1}) + 1$ and $a(v_{2j}) + 1$). It is easy to check that this incidentor coloring is indeed (k, k)-coloring.

It remains to show that $\chi_{0,1}^{list}(G) = 2$. We use Vising's theorem[1] mentioned above. According to this theorem for any lists of cardinality 2 there exists a list 0-coloring of a graph G. Since in this case the list of admissible colors for any arc is $\{a, a+1\}$ for some a a list 0-coloring automatically is a $(0, 1)$-coloring. Theorem 2 is proved.

Using Vising's theorem[1] we can also prove a general bound on the list incidentor (k, l)-chromatic number.

Theorem 3. *For any multigraph of even degree Δ holds:*

$$\chi_{k,k+\Delta-1}^{list}(G) \leq k + \Delta.$$

Proof. By Vising's theorem $\chi_k^{list}(G) \leq k+\Delta$. Since the absolute value of the difference of any two colors from the interval $[a, a+k+\Delta-1]$ is at most $k + \Delta - 1$ any k-coloring automatically is a $(k, k+\Delta)$-coloring. Theorem 3 is proved.

Note that for graphs of odd degree Δ in the same way we obtain the bound $\chi_{k,k+\Delta}^{list}(G) \leq k + \Delta + 1$.

Theorems 1 and 3 may be generalized to the following conjecture:

Conjecture 1. *For any multigraph of degree Δ and $l \leq k + \Delta - 1$ holds:*

$$\chi_{k,l}^{list}(G) \leq 2k + 2\Delta - l - 1.$$

In the next section this conjecture is proved for multigraphs of degree 4.

2 Multigraphs of degree 4

From Theorems 1 and 3 we get that for any multigraph G of degree 4 we have $\chi^{list}_{k,k}(G) \leq k+7$ and $\chi^{list}_{k,k+3}(G) \leq k+4$. In order to prove Conjecture 1 it remains to show that $\chi^{list}_{k,k+2}(G) \leq k+5$ and $\chi^{list}_{k,k+1}(G) \leq k+6$. We consider this cases separately. Without loss of generality we assume that G is regular (as adding new arcs to irregular graph will not make a coloring process any easier).

Let us first introduce some notations. By Petersen's theorem any regular multigraph of degree 4 can be decomposed into the union of two 2-factors. Denote them by F_1 and F_2. All incidentors of this 2-factors we partition into even and odd in such a way that every arc contains one even and one odd incidentor and every vertex is adjacent to one even and one odd incidentor from both 2-factors. Let v is an arbitrary vertex in G. It is adjacent to four incidentors. Denote them by $i_1(v), i_2(v), i_3(v), i_4(v)$ such that $i_1(v), i_2(v) \in F_1$, $i_3(v), i_4(v) \in F_2$, and $i_1(v), i_3(v)$ are odd, and $i_2(v), i_4(v)$ are even.

Theorem 4. *For any multigraph G of degree 4 the following bound holds:*

$$\chi^{list}_{k,k+2}(G) \leq k+5.$$

Proof. Let an arc e have a list $[a, a+k+4]$. Call the colors $a, a+1$ and $a+2$ *essential* for the initial incidentor of the arc e and colors $a+k+2, a+k+3$ and $a+k+4$ for the final incidentor of the arc e. Note that if incidentor i is colored by an essential color than the incidentor i' mated to i has 3 options of proper coloring that satisfy the demand on the difference of colors. Also colors $a+2$ and $a+k+2$ are called *middle* for the initial and the final incidentor of the arc e respectively. Given an incidentor i denote by $s(i)$ its middle color and by $M(i)$ — the set of its essential colors. For any vertex v put $M'(i_1(v)) = M(i_1(v)) \setminus \{s(i_4(v))\}$. It is clear that $M'(i_1(v))$ contains at least two colors. Call one of them the *main color* and the other one the *reserve color*. We construct the coloring in 4 steps.

Step 1. Color all odd incidentors of F_1 by the main colors.

Step 2. Consider an arc $e = uv \in F_2$ with the list $[c, c+k+4]$, and let i be its odd incidentor. Without loss of generality i is initial, i.e. $i = i_3(u)$. Denote by a and b the colors of incidentors $i_1(u)$ and $i_1(v)$ respectively. By the definition of $M'(i_1(v))$ we have $b \neq c+k+2$. Consider two cases.

a) If $b \notin \{c+k, c+k+1\}$, we color i by the color c. In the same time if $a = c$, we recolor the incidentor $i_1(u)$ by the reserve color.

b) If $b \notin \{c+k+3, c+k+4\}$, we color i by the color $c+2$. In the same time if $a = c+2$, we recolor the incidentor $i_1(u)$ by the reserve color.

Step 3. Color all even incidentors in F_1 by a free admissible color. For any incidentor $i_2(v)$ such color exists, because the incidentor mated to it is colored by an essnetial color, and the vertex v is adjacent only with two colored incidentors.

Step 4. We need to show that after Steps 1-3 any uncolored incidentor $i = i_4(u)$ has a free admissible color. Indeed, on Step 1 the incidentor $i_1(u)$ is colored by a color $b \neq s(i)$. On Step 2 the incidentor i' is colored in such a way that for the incidentor i there are three free admissible colors that are not equal to b. Besides, on Step 2 we color the incidentor $i_3(u)$; during this operation the color of $i_1(u)$ either remains the same, or it is changed by the reserve color, but in the latter case $i_3(u)$ is colored by b. Anyway, after Step 2 there are at least two colors left for the incidentor $i_4(u)$. It is clear that after Step 3 at most one of them is used for coloring incidentor $i_2(u)$. Thus, we can color all even incidentors in F_2 by admissible colors. Theorem 4 is proved.

Theorem 5. *For any multigraph G of degree 4 the following bound holds*

$$\chi^{list}_{k,k+1}(G) \leq k+6.$$

Proof. Let an arc e have a list $[a, a+k+5]$. Call the colors from the interval $[a, a+4]$ *essential* for the initial incidentor of the arc e, and colors from the interval $[a+k+1, a+k+5]$ for the final incidentor of the arc e. It is clear that if incidentor i is colored by an essential color, then the incidentor i' mated to it has two options of proper colorings. Denote the set of essential colors of the incidentor i by $M(i)$.

We find a coloring in three steps.

Step 1. Color all odd incidentors in F_1 by essential colors in such a way that for any even incidentor in F_1 two free admissible colors would left. Let us show that this is possible. Consider an arbitrary arc $e = uv \in F_1$ and let its incidentor $i = i_1(u)$ be odd (the case when $i = i_1(v)$ is odd is similar). Denote by a the color of the incidentor $i_1(v)$, by b — the color of the incidentor mated to $i_2(u)$ (if these incidentors are already colored). Then, obviously, the incidentor i cannot be colored by the colors $a-k, a-k-1$, and also by the colors $b+k, b+k+1$, if $i_2(u)$ is the final inicidentor, or by the colors $b-k, b-k-1$, if $i_2(u)$ is the initial inicidentor. But since $|M(i)| = 5$, there is always an essential color for the incidentor i, that satisfies the demands.

For any even incidentor i in F_1 denote by $M'(i)$ the set of free admissible colors for this inicdentor after Step 1. By the construction, $|M'(i)| = 2$.

Step 2. On this step we color incidentors in F_2. First for any vertex v we delete from $M(i_3(v))$ the color of the incidentor $i_1(v)$ and both colors from $M'(i_2(v))$. We are left with some set of essential colors of the incidentor $i_3(v)$, that we denote by $M'(i_3(v))$. Each of these colors provides two coloring options for the incidentor $i = i'_3(v)$; denote the set of such options as $P(i)$. It is clear that $|P(i)| \geq 3$, and the equality is achieved only in case when $M'(i_3(v)) = \{x, x+1\}$ for some color x.

Choose an arbitrary component C in F_2. Consider three following cases:

Case 1. Assume that there are an arc $e = uv$ with an odd incidentor $i = i_3(u)$ in C and a color $x \in M'(i)$ such that neither $x+k$, nor $x+k+1$ coincides with the color of the incidentor $i_1(v)$. Then we color the incidentor i by the color x and the incidentor $i_4(u)$ by the color from $P(i_4(u))$, that coincides neither with x, nor with the color of $i_1(u)$; after that we color mated incidentor $j = i'_4(u)$ by the suitable color from $M'(j)$ (which exists by the definition of $M'(j)$), and so on, following the cycle C. Thus, for each arc from C (except the arc e) we first color its even incidentor i using a color from $P(i)$, that doesn't coincide with the colors of incidentors, adjacent with i, and then we color the incidentor i' mated with i by a suitable color from $M'(i')$. Finally, the even incidentor $i_4(v)$ of the arc e is colored by the color from $\{x+k, x+k+1\}$, that is not equal to the color of the incidentor $i_3(v)$.

Case 2. Assume that there are an arc $e = uv$ with an odd incidentor $i = i_3(v)$ in C and a color $x \in M'(i)$, such that neither $x-k$, nor $x-k-1$ coincides with the color of the incidentor $i_1(u)$. This case is similar to the prevoius one.

Case 3. Assume that for every odd incidentor i and every color in $M'(i)$ at least one of two possible ways of coloring the incidentor i' uses the color of the odd incidentor j of 2-factor F_1 adjacent to i'. This may happen only if $P(i') = \{x-1, x, x+1\}$, the incidentor j is colored by the color x, and the colors in $M'(i)$ are consecutive.

Consider an arbitrary arc $e = uv$. Without loss of generality, the initial incidentor $i = i_3(u)$ is odd and $M'(i) = \{x, x+1\}$. By the assumption, the incidentor $i_1(v)$ is colored by the color $x+k+1$, so there are two ways to color arc e: either $f(e) = (x, x+k)$, or $f(e) = (x+1, x+k+2)$. The same is true for any arc in C. Since colors in $M(i_3(v))$ are also consecutive, at least one of them equals neither to $x+k$, nor to $x+k+2$. So we color the cycle C starting with the arc containig $i_3(v)$; this arc we color in such a way that the color $i_3(v)$ equals neither to $x+k$, nor to $x+k+2$. Continue the coloring of C moving from v to u. Then for any arc in the cycle C, including the last arc e, one of two coloring options is proper.

Repeat this procedure for all other components of F_2.

Step 3. Color all even incidentors in F_1 by free admissible colors. Make sure that this is possible. Indeed, consider an arbitrary vertex v. After Step 1 the set of free admissible colors $M'(i_2(v))$ contains exactly 2 colros. On Step 2 the odd incidentor $i_3(v)$ is colored by the color from $M'(i_3(v))$, which by definition cannot contain colors from $M'(i_2(v))$. But then after deleting from $M'(i_2(v))$ the color of the incidentor $i_4(v)$ at least one free admissible color for the inicidentor $i_2(v)$ remains at our disposal. Theorem 5 is proved.

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