Randomness deficiencies

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Abstract

The notion of random sequence was introduced by Martin-Löf in [4]. In the same article he defined the so-called randomness deficiency function that shows how close are random sequences to non-random (in some natural sense). Other deficiency functions can be obtained from the Levin-Schnorr theorem, that describes randomness in terms of Kolmogorov complexity. The difference between all of these deficiencies is bounded by a logarithmic term (proposition 1). In this paper we show (theorems 1 and 2) that the difference between some deficiencies can be as large as possible.

1 Introduction

Classical probability theory cannot deal with individual random objects, such as binary sequences or points on the real line: each sequence or point has measure zero (with respect to the uniform measure). However our intuition says that the sequence of zeros (and any other computable sequence) is not random, while the result of tossing a coin is random. Martin-Löf in [4] tried to formalize this statement. He used an algorithmic approach to define random binary sequences.

Martin-Löf random sequences have many nice properties: adding, deleting or changing finitely many bits doesn't change randomness; random sequences satisfy the law of large numbers; computable permutations preserve randomness. So if the sequence ω is random, the sequence $\omega' = 0^{1000000000}\omega$ (billion of zeros concatenated with ω) is also random. But intuitively ω' is "less random". We can make this arguement formal using a randomness deficiency function d: this function is finite on random sequences and infinite on non-random sequences. If $d(\omega') \geq d(\omega)$ we say that ω' is less random than ω. It turns out that there are some natural types of deficiency functions that have similar properties to the so-called finite deficiency (the difference between the length of the string and its Kolmogorov complexity). For example, adding n zeros to the sequence increases randomness deficiency by $n + O(\log n)$. Using this fact one can reformulate statements about random sequences in terms of the deficiency functions to look for the connections between algorithmic randomness and Kolmogorov complexity theories.

In this paper we consider several deficiency functions: the first was introduced by Martin-Löf (definition 3), the others appear from the Levin-Schnorr's criterion of randomness in terms of different types of Kolmogorov complexity: the prefix-free complexity (1) and the a priori complexity (definition 13). The difference between all of the deficiencies is not greater than $(1+\varepsilon)\log d$ (up to a constant, for all $\varepsilon > 0$) (proposition 1), where d is one of the deficiency functions. We show that the difference between some of the deficiencies can be greater than $\log d$. For example, some of the deficiency functions (given in the exponential scale) are integrable, while the others are not and that is the reason of the difference (theorem 1). To differ the integrable deficiencies we construct a special rarefied set of intervals in the Cantor space (theorem 2).

Notation

The set of all infinite binary sequences is called the Cantor space and is denoted by Ω. An interval in the Cantor space is a set of extensions of some string x, it is denoted by $[x]$. The set of all binary strings is denoted by $\mathbb{B}^*$. The length of the string x is denoted by $|x|$. We write $y \prec x$ if y is a prefix of x. $\mathbb{I}_S$ is the indicator function of the set S. log means binary logarithm. Notation $f <^+ g \quad (f <^* g)$ means that there exists a constant c such that for all $x \;\; f(x) < c + g(x) \quad (f(x) < cg(x))$.

2 Preliminaries

One can find all of the notions and statements of this section in [1] and [2].

Definition 1. A measure μ over Ω is called computable, if there exists a Turing machine that from each string x and rational $\varepsilon > 0$ returns an ε-approximation of the value $\mu([x])$.

The collection of intervals in the Cantor space forms a base for its standard topology. We will talk about closed and open sets relative to this topology.

Definition 2. Let μ be a computable measure. A nested sequence of open sets $\{V_n\}$ is called a Martin-Löf test with respect to μ if:

1) $\{V_n\}$ is uniformly effectively open, that is there exists a Turing machine that for each input k enumerates the set V_k.

2) $\mu(V_n) \leq 2^{-n}$ for each n.

Definition 3. Let $\{V_n\}$ be a Martin-Löf test with respect to a computable measure μ. Function $d_{\mu;\{V_n\}}(\omega) = \max\{k : \omega \in V_k\}$ is called a randomness deficiency of ω with respect to the test $\{V_n\}$.

Lemma 1. *For every computable measure μ there exists a Martin-Löf test $\{U_n\}$ with respect to a computable measure μ such that for any Martin-Löf test $\{V_n\}$ with respect to μ there exist a constant c such that for all sequences ω*

$$d_{\mu;\{U_n\}}(\omega) \geq d_{\mu;\{V_n\}}(\omega) - c$$

Proof. We can enumerate all Martin-Löf tests $\{U_n^j\} : U_1^j \supset U_2^j \supset \cdots$ and construct a new test:

$$\begin{aligned} U_1 = U_2^1 \cup U_3^2 \cup \ldots \supset \cdots \supset U_2 = U_3^1 \cup U_4^2 \cup \ldots \supset \cdots \supset \\ \supset \cdots \supset U_n = U_{n+1}^1 \cup U_{n+2}^2 \cup \ldots \supset \cdots \end{aligned}$$

The new deficiency $\mathbf{d}_\mu$ is not less than $d_{\mu;\{U_n^j\}} - j$. □

The deficiency function $\mathbf{d}_\mu$ was defined by Martin-Löf in [4]. In the same article he introduced the following notion of randomness:

Definition 4. Let μ be a computable measure. A sequence $\omega \in \Omega$ is called Martin-Löf random with respect to μ if $\mathbf{d}_\mu(\omega) < \infty$.

There are some other types of deficiency functions. To show the relations between them, we need to reformulate the definition of $\mathbf{d}_\mu$. First we define the so-called lower semicomputable functions.

Definition 5. A function $t : \Omega \to \mathbb{R}$ is called lower semicomputable if there exists a machine that by rational r enumerates the set of intervals $\{\omega : t(\omega) > r\}$ (so this set should be open).

Let's note the following property of $\mathbf{d}_\mu$: the function $\mathbf{t}_\mu = 2^{\mathbf{d}_\mu}$ is probability bounded, that is

$$\mu\{\mathbf{t}_\mu(\omega) > c\} \leq \frac{1}{c}$$

for rational numbers c. Moreover, $\mathbf{t}_\mu$ is the largest (up to a multiplicative constant) among all lower semicomputable probability bounded functions (the sets $V_n = \{t(\omega) > 2^n\}$ form a Martin-Löf test). Therefore we can define the function $\mathbf{d}_\mu$ as logarithm of the largest lower semicomputable probability bounded function and from now we denote this function as $\mathbf{d}_\mu^P$ (and $\mathbf{t}_\mu$ as $\mathbf{t}_\mu^P$).

To define other deficiency functions we need the following notion:

Definition 6. Function $f : \Omega \to \mathbb{Q}$ is called basic if its value on every sequence ω is determined by some finite prefix of ω.

By compactness of Ω there exist finitely many intervals where basic function is constant, and the union of these intervals is Ω. Therefore basic functions are constructive objects and we can consider computable sequences of basic functions.

The following lemma gives the equivalent definition of lower semicomputable functions.

Lemma 2. *Function $t : \Omega \to \mathbb{R}$ is lower semicomputable iff it is a limit of increasing computable sequence of basic functions.*

Proof. If the function t is lower semicomputable then t is a supremum of basic functions $t_{n;k}(\omega) = n\mathbb{I}_{A_k}(\omega)$, where A_k is a set of intervals produced after k steps of enumeration of $\{t_\mu(\omega) > n\}$. Supremum is a limit of maximums and maximum over the finite set of basic functions is also a basic function. If t is a limit of increasing computable sequence of basic functions t_n then for given r we can produce intervals where $t_j > r$ for all j. □

If the function is integrable and its integral is less than 1 it is probability bounded (by Markov's inequality). We call these functions expectation bounded. There exists maximal (up to a multiplicative constant) lower semicomputable expectation bounded function $\mathbf{t}_\mu^E$: we can enumerate all probability bounded functions (with respect to μ); the integral of such function is a limit of integrals of basic functions, so if it is greater than 1 we always know it after finitely many steps of computation. If the integral is greater than 1, we decrease the values of basic functions to make it less than 1. The sum of these new functions with weights 2^{-n} is the maximal lower semicomputable expectation bounded function.

Definition 7. Let μ be a computable measure. The expectation bounded deficiency is the function

$$\mathbf{d}_\mu^E(\omega) = \log \mathbf{t}_\mu^E(\omega)$$

The following proposition shows that the difference between $\mathbf{d}_\mu^p$ and $\mathbf{d}_\mu^E$ is not large.

Proposition 1. *Let μ be a computable measure and $\varepsilon > 0$. Then*

$$\boldsymbol{d}_\mu^E \leq^+ \boldsymbol{d}_\mu^P \leq^+ \boldsymbol{d}_\mu^E + (1+\varepsilon)\log \boldsymbol{d}_\mu^E$$

Proof. The first part follows from Markov's inequality. To prove the second part, let's consider a function $\mathbf{t}_\mu^P \log^{-1-\varepsilon} \mathbf{t}_\mu^P$. Its integral does not exceed

$$\sum_n \int_{A_n} \mathbf{t}_\mu^P(\omega) \log^{-1-\varepsilon} \mathbf{t}_\mu^P(\omega) d\mu(\omega) \leq \sum_n 2n^{-1-\varepsilon}$$

where $A_n = \{2^n \leq \mathbf{t}_\mu^P < 2^{n+1}\}$, so this integral is finite. Therefore

$$\mathbf{d}_\mu^P \leq^+ \mathbf{d}_\mu^E + (1+\varepsilon)\log \mathbf{d}_\mu^P \leq^+ \mathbf{d}_\mu^E + (1+\varepsilon)\log \mathbf{d}_\mu^E$$

□

The deficiency function $\mathbf{d}_\mu^E$ can be described in terms of prefix-free Kolmogorov complexity (see, for example, [2]). We will briefly describe this construction. At first we define the discrete analogues of basic and lower semicomputable functions.

Definition 8. Function $f : \mathbb{B}^* \to \mathbb{Q}$ is called basic if its support is finite.

Definition 9. Function $f : \mathbb{B}^* \to \mathbb{R}$ is called lower semicomputable if it is a limit of increasing computable sequence of basic functions.

Definition 10. Lower semicomputable function $m : \mathbb{B}^* \to [0,\infty)$ such that $\sum_x m(x) \leq 1$ is called discrete lower semicomputable semimeasure.

Let's denote the prefix-free Kolmogorov complexity of a string x as $K(x)$. The function $\mathbf{m}(x) = 2^{-K(x)}$ is called the discrete a priori probability. The famous coding theorem (see, for example, [2]) states that this function is the largest (up to a multiplicative constant) among all discrete lower semicomputable semimeasures.

It can be shown (see, for example, [1]) that

$$\mathbf{t}_\mu^E(\omega) =^* \sum_n \frac{\mathbf{m}(\omega_{1\ldots n})}{\mu([\omega_{1\ldots n}])} =^* \sup_n \frac{\mathbf{m}(\omega_{1\ldots n})}{\mu([\omega_{1\ldots n}])}$$

In the logarithmic scale:

$$\mathbf{d}_\mu^E(\omega) =^+ \sup_n\{-\log \mu([\omega_{1\ldots n}]) - K(\omega_{1\ldots n})\} \tag{1}$$

This result is due to Gacs (see [5]). The value in the right part of 1 is finite iff the sequence is random. It was first shown by Schnorr and Levin independently in [6] and [7]. Informally, the sequence is random iff its initial segments are incompressible. The equation 1 also shows that if one adds n zeros to the sequence then the randomness deficiency (probability or expectation bounded) increases by at most $n + O(\log n)$.

The Schnorr-Levin theorem can be formulated in terms of the so-called a priori complexity. To define it we need the notion of continuous a priori probability.

Definition 11. Lower semicomputable function $a : \mathbb{B}^* \to [0, \infty)$ such that $\sum_{x \in S} a(x) \leq 1$ for every prefix-free set S is called continuous lower semicomputable semimeasure.

We can enumerate all continuous lower semicomputable semimeasures and consider a semimeasure $\mathbf{a}(x) = \sum_j a_j(x)\mathbf{m}(a_j)$. This semimeasure is also continuous and lower semicomputable, and it is the largest (up to a multiplicative constant) in this class of semimeasures. We will call $\mathbf{a}(x)$ the continuous a priori probability.

Definition 12. The value $KA(x) = -\log \mathbf{a}(x)$ is called the a priori complexity of x.

The Schnorr-Levin theorem for the a priori complexity states that the sequence ω is random iff $\sup_n\{-\log \mu([\omega_{1\ldots n}]) - KA(\omega_{1\ldots n})\}$ is finite. Moreover, supremum can be replaced by lim sup or lim inf. Using this theorem we can define other types of deficiency functions.

Definition 13. Let μ be a computable measure. We will consider functions

$$\mathbf{d}_\mu^A(\omega) = \sup_n\{-\log \mu([\omega_{1\ldots n}]) - KA(\omega_{1\ldots n})\}$$
$$\mathbf{d}_\mu^{\limsup A}(\omega) = \limsup_n\{-\log \mu([\omega_{1\ldots n}]) - KA(\omega_{1\ldots n})\}$$
$$\mathbf{d}_\mu^{\liminf A}(\omega) = \liminf_n\{-\log \mu([\omega_{1\ldots n}]) - KA(\omega_{1\ldots n})\}$$

and call them a priori randomness deficiencies.

Each continuous lower semicomputable semimeasure can be represented as a probability distribution on the initial segmets of outputs of some probabilistic machine that prints bits one after another and does not have to stop (see, for example, [2]). That is for each $a(x)$ there exists a machine A such that

$$a(x) = \mathbb{P}\{\text{the output of } A \text{ begins on the string } x\}$$

Informally, the Schnorr–Levin theorem states that the sequence ω is random iff the probability of getting the initial segments $\omega_{1\ldots n}$ using a probabilistic machine cannot be much greater than getting it from a random generator (with the distribution μ). The deficiency functions from the definition 13 show the difference between logarithms of these probabilities.

One can use supermartingales to define the deficiencies $\mathbf{d}_\mu^A$, $\mathbf{d}_\mu^{\limsup A}(\omega)$, $\mathbf{d}_\mu^{\liminf A}(\omega)$.

Definition 14. Let μ be a measure on Ω and let M be a function of binary strings.

If $\mu([x])M(x) = \mu([x0])M(x0) + \mu([x1])M(x1)$ the function M is called a martingale.

If $\mu([x])M(x) \geq \mu([x0])M(x0) + \mu([x1])M(x1)$ the function M is called a supermartingale.

If $\mu([x])M(x) \leq \mu([x0])M(x0) + \mu([x1])M(x1)$ the function M is called a submartingale.

If martingale (or sub/supermartingale) is not bounded on the initial segments of the sequence ω we say that it wins on ω.

If μ is computable, the supermartingale $\mathbf{M}(x) = \frac{\mathbf{a}(x)}{\mu([x])}$ is the largest (up to a multiplicative constant) among all lower semicomputable supermartingales. Supermartingale $\mathbf{M}(x)$ wins on all non-random sequences and does not win on random sequences.

The deficiency $\mathbf{d}_\mu^A(\omega)$ is a supremum of $\mathbf{M}(\omega_{1\ldots n})$, the deficiencies $\mathbf{d}_\mu^{\limsup A}(\omega)$ and $\mathbf{d}_\mu^{\liminf A}(\omega)$ are respectively limsup and liminf of $\mathbf{M}(\omega_{1\ldots n})$.

Now we are going to show the relations between the deficiencies.

Proposition 2.

$$\boldsymbol{d}_\mu^E \leq^+ \boldsymbol{d}_\mu^{\liminf A}$$

Proof. We need to construct some continuous lower semicomputable semimeasure a. Once the approximation to $\mathbf{m}(x)$ increases by ε we do the following:

1)increase the value of a by ε on prefixes of x

2)increase the value of a by $\varepsilon\mu([y])/\mu([x])$ on the extensions y of x.

If $\mathbf{d}_\mu^E = R$ there exists a string x such that

$$-\log \mu([x]) - K(x) =^+ R$$

and ω is the extension of x. If $n > |x|$, the logarithm of a is:

$$\log a(\omega_{1\ldots n}) \geq -K(x) + \log \mu([\omega_{1\ldots n}]) - \log \mu([x])$$

Therefore

$$\begin{aligned}\mathbf{d}_\mu^{\liminf A}(\omega) &\geq^+ \liminf_n \{-\log \mu([\omega_{1\ldots n}]) + \log a(\omega_{1\ldots n})\} \geq \\ &\geq \liminf_n \{-\log \mu([x]) - K(x)\} = -\log \mu([x]) - K(x) =^+ \mathbf{d}_\mu^E\end{aligned}$$

The case $\mathbf{d}_\mu^E = \infty$ can be considered in the same way. $\square$

Proposition 3.
$$\boldsymbol{d}_\mu^A \leq^+ \boldsymbol{d}_\mu^P$$

Proof. It is sufficient to show that $\mu\{2^{\mathbf{d}_\mu^A}(\omega) > 2^c\} \leq 2^{-c}$ for all rational c. Let's fix c and consider a set of strings

$$S = \{x : \frac{\mathbf{a}(x)}{\mu([x])} > 2^c,\ \forall y \prec x\ \ \frac{a(y)}{\mu([y])} \leq 2^c\}$$

It is evident that $\omega \in \cup_{x\in S}[x]$ iff $\mathbf{d}_\mu^A(\omega) > c$. The set S is prefix-free, so

$$\mu\{2^{\mathbf{d}_\mu^A}(\omega) > 2^c\} = \sum_{x\in S} \mu([x]) < \sum_{x\in S} \frac{a(x)}{2^c} \leq 2^{-c}$$

□

Combining the results of Propositions 1, 2 and 3 we can write down the following chain of inequalities:

$$\mathbf{d}_\mu^E \leq^+ \mathbf{d}_\mu^{\liminf A} \leq^+ \mathbf{d}_\mu^{\limsup A} \leq^+ \mathbf{d}_\mu^A \leq^+ \mathbf{d}_\mu^P \leq^+ \mathbf{d}_\mu^E + (1+\varepsilon)\log \mathbf{d}_\mu^E$$

The natural question is about the difference between these deficiencies.

3 New results

Now we are going to show the relations between deficiency functions. Proposition 4 is an effective version of Doob's martingale convergence theorem (see, for example, [8]) and can be easily obtained from it. Theorems 1 and 2 require lemma 3. This lemma can be easily proved using standard techniques from calculus.

Definition 15. If the sequence ω is random relative to the oracle $0'$ it is called 2-random.

Proposition 4. *Let μ be a computable measure. If ω is 2-random (with respect to μ), then* $\boldsymbol{d}_\mu^{\limsup A}(\omega) = \boldsymbol{d}_\mu^{\liminf A}(\omega)$

Proof. Given rational numbers $\beta > \alpha > 0$ we can construct a $0'$-computable supermartingale M_α^β that wins on such sequences ω that the supermartingale $\mathbf{M}$ infinitely many times becomes smaller than α and greater than β on the initial segments of ω. Using the oracle we compute the values of $\mathbf{M}$ and if $\mathbf{M}(x) < \alpha$ the values $M_\alpha^\beta(z)$ are equal to $\mathbf{M}(z)$ on such extensions z of x that $\mathbf{M}(z) \leq \beta$. When we find such extension y that $\mathbf{M}(y) > \beta$ we just save the capital ($M_\alpha^\beta(yw) = M_\alpha^\beta(y)$) until we find some new string x with small $\mathbf{M}(x)$. On the segments from x to y the value of M_α^β increases by $\frac{\beta}{\alpha}$ times. The sum of all M_α^β with weights $\mathbf{m}(\alpha,\beta)$ is a $0'$-lower semicomputable supermartingale, so it is finite on 2-random sequences. □

Corollary 1. *Let μ be a computable measure. Then $2^{\mathbf{d}_\mu^{\limsup A}}$ is the integrable function with respect to μ.*

Proof. By Fatou's lemma:

$$\int_\Omega \liminf_n \mathbf{M}(\omega_{1\ldots n}) d\mu(\omega) \leq \liminf_n \int_\Omega \mathbf{M}(\omega_{1\ldots n}) d\mu(\omega) = \liminf_n \sum_{|x|=n} \mathbf{a}(x) \leq 1$$

$\mathbf{d}_\mu^{\limsup A} = \mathbf{d}_\mu^{\liminf A}$ almost everywhere, therefore $2^{\mathbf{d}_\mu^{\limsup A}}$ is integrable. □

The greater deficiencies are not integrable (in the exponential scale). To show that $2^{\mathbf{d}_\mu^A}$ is not integrable we need the following easy lemma from calculus:

Lemma 3. *If $c_k \geq 0$ and $\sum_{k=1}^\infty c_k < \infty$ and $R_k := \sum_{n=k+1}^\infty c_n > 0$, then*

$$\sum_{k=1}^\infty \frac{c_k}{R_k \log \frac{1}{R_K}} = \infty$$

Proof. At first we will prove that

$$\sum_{k=1}^\infty \frac{c_k}{R_k} = \infty$$

Denote $z_k = \frac{c_k}{R_k}$. It is evident that

$$z_k = \frac{R_{k-1} - R_k}{R_k} = \frac{R_{k-1}}{R_k} - 1$$

Therefore

$$\frac{1}{R_k} = \frac{1}{R_0} \prod_{n=1}^k (1 + z_n)$$

If we take the logarithm from both parts, we get

$$\log \frac{1}{R_k} = \log \frac{1}{R_0} + \sum_{n=1}^k \log(1 + z_n) \leq^* \sum_{n=1}^k z_n \tag{2}$$

The left part tends to infinity, so the sum $\sum_{n=1}^{\infty} z_n$ is infinite. To prove the lemma we need to show that $\sum_{k=1}^{\infty} \frac{z_k}{\log \frac{1}{R_k}} = \infty$. Using 2 we get:

$$\sum_{k=1}^{\infty} \frac{z_k}{\log \frac{1}{R_k}} \geq^* \sum_{k=1}^{\infty} \frac{z_k}{\sum_{n=1}^{k} z_n}$$

Denote $S_k = \sum_{n=1}^{k} z_n$ and $b_k = \frac{z_k}{S_k}$. It is sufficient to show that if the series $\sum_{n=1}^{\infty} z_n$ does not converge then the series $\sum_{n=1}^{\infty} b_n$ also does not converge. We will do it in the same way as the first part of the proof of the lemma:

$$b_k = \frac{S_{k+1} - S_k}{S_k} = \frac{S_{k+1}}{S_k} - 1$$

Therefore

$$S_{k+1} = S_1 \prod_{n=1}^{k} (1 + b_n)$$

If we take the logarithm from both parts we get

$$\log S_k = \log S_1 + \sum_{n=1}^{k} \log(1 + b_n) \leq^* \sum_{n=1}^{k} b_n$$

The left part tends to infinity, so the sum $\sum_{n=1}^{\infty} b_n$ is infinite. $\square$

Recall the definition of atomic measures.

Definition 16. If the measure μ on Ω is positive on some sequence, we will say that μ is an atomic measure.

Now we are ready to prove two statements about the difference between $\mathbf{d}^A$ and other deficiencies.

Theorem 1. *Let μ be a computable non-atomic measure. For all c there exists ω such that*

$$\boldsymbol{d}_\mu^{\limsup A}(\omega) < \boldsymbol{d}_\mu^A(\omega) - \log \boldsymbol{d}_\mu^A(\omega) - c$$

Proof. It is sufficient to prove that the function $q = 2^{\mathbf{d}_\mu^A - \log \mathbf{d}_\mu^A}$ is not integrable with respect to μ. We will construct some deterministic (but formally probabilistic) machine f. At each step, after f has printed the string of bits x of length k, f computes measures of $[x0]$ and $[x1]$, and then prints a bit b if $\mu[xb] > \frac{1}{3}\mu[x]$ (if the both bits are suitable, let f print 0). Denote the interval $[xb] = B_k$ if at the k-th step f prints a bit b, and $C_k = B_{k-1} - B_k$. The measure μ is non-atomic, hence

$$\mu B_k = \sum_{n=k+1}^{\infty} C_n$$

The intervals C_k are disjoint, so $\sum_k C_k \leq 1$. By lemma 3:

$$\sum_{k=1}^{\infty} \frac{\mu C_k}{\mu B_k \log \frac{1}{\mu B_k}} = \infty$$

Let's denote

$$a_f(x) = \mathbb{P}\{\text{the output of } f \text{ begins on the string } x\}$$

and

$$t_f(\omega) = \sup_n \frac{a_f(\omega_{1\dots n})}{\mu([\omega_{1\dots n}])}$$

The function $\frac{x}{\log x}$ is monotone for large enough x, therefore by the universality

$$q \geq^* \frac{t_f}{\log t_f}$$

It is easy to see that

$$\frac{t_f}{\log t_f}(\omega) = \sum_{k=1}^{\infty} \frac{\mathbb{I}_{C_{k+1}}}{\mu B_k \log \frac{1}{\mu B_k}}(\omega)$$

Recall that $\mu B_k \geq \mu B_{k+1} > \frac{1}{3}\mu B_k$

$$\int_\Omega q(\omega) d\omega \geq^* \int_\Omega \frac{t_f}{\log t_f}(\omega) d\omega \geq \sum_{k=1}^{\infty} \frac{\mu C_{k+1}}{\mu B_k \log \frac{1}{\mu B_k}} >$$
$$> \frac{1}{3} \sum_{k=1}^{\infty} \frac{\mu C_{k+1}}{\mu B_{k+1} \log \frac{1}{\mu B_{k+1}}} = \infty$$

□

The next theorem requires some technical constructions in general case, so at first we will prove it in the case of the uniform measure to show the idea.

Theorem 2. *Let μ be a computable non-atomic measure. For all c there exists ω such that*

$$\boldsymbol{d}_\mu^A(\omega) < \boldsymbol{d}_\mu^P(\omega) - \log \boldsymbol{d}_\mu^P(\omega) - c$$

Proof of the uniform case. The main idea is that one cannot win 50$ after 5 tosses of a coin if he starts with 1$.

Let's consider a function $g = \sum_k 2^{2k-1}\mathbb{I}_{[0^k1^k]}(\omega)$. It is a lower semicomputable probability bounded function. Let's prove the theorem by contradiction. Assume that there exists a constant c such that for all ω

$$\mathbf{t}^A_\mu(\omega) \geq 2^{-c}\frac{g}{\log g}(\omega)$$

That means that there exists a prefix-free set of binary strings w^k_l such that $\cup_l[w^k_l] \supset 0^k1^k$ and

$$\mathbf{a}(w^k_l)2^{|w^k_l|} \geq 2^{-c}\frac{2^{2k-1}}{2k-1}$$

For k large enough

$$|w^k_l| \geq -c - \log(2k-1) + 2k - 1 + KA(w^k_l) > k+1$$

So $[w^k_l] \subset [0^k1]$. Hence the set $\{w^k_l\}_{k,l}$ is prefix-free. Consider the following chain of inequalities:

$$1 \geq \sum_k\sum_l \mathbf{a}(w^k_l) \geq \sum_k\sum_l 2^{-c-|w^k_l|}\frac{2^{2k-1}}{2k-1} \geq^+$$
$$\geq^+ \sum_k 2^{-|0^k1^k|}\frac{2^{2k-1}}{2k-1} = \sum_k \frac{1}{2(2k-1)} = \infty$$

This contradiction proves the theorem. □

Proof of the general case. Now we replace the intervals $[0^k1]$ and $[0^k1^k]$ by C_k and D_k (see below) respectively. We cannot make the measures of D_k very small, because it decreases g, but they also cannot be large, because g should be probability bounded. We will find suitable sets $\{C_k\}$ and $\{D_k\}$ that satisfy all of the conditions.

Let's consider the intervals B_k and C_k from theorem 1. The series $\sum \mu(C_k)$ is computable, therefore the ordering τ of $\{C_k\}$ (the first element of the ordering has maximal measure over $\{C_k\}$, the second has maximal measure over the rest of $\{C_k\}$, etc.) is also computable. Denote the elements of this ordering by $\mathbf{C}_k$ and consider $z_k = -\frac{3}{\log \mu\mathbf{C}_k}$. The sequence $S_k = 1 + \sum_{j\leq k} z_k$ is computable. Let's show that $S_k \to \infty$:

Recall that

$$\sum_k \frac{\mu C_{k+1}}{\mu B_k \log\frac{1}{\mu B_k}} = \infty$$

The function $\frac{x}{\log x}$ is monotone for large enough x, therefore

$$\sum_k \frac{3}{\log \frac{1}{\mu C_k}} = 3 \sum_k \frac{\mu C_k}{\mu C_k \log \frac{1}{\mu C_k}} \geq 3 \sum_k \frac{\mu C_{k+1}}{\mu B_k \log \frac{1}{\mu B_k}} = \infty$$

Now we are going to construct the set of intervals $D_k \subset C_k$ with such property:

$$\frac{1}{3}(\mu C_k)^{S_{\tau(k)}} < \mu D_k < (\mu C_k)^{S_{\tau(k)}} \tag{3}$$

Let x_k be a string such that $[x_k] = C_k$. We compute $\mu([x_k 0])$ and $\mu([x_k 1])$ and choose the next bit b if $\mu[x_k b] > \frac{1}{3}\mu[x_k]$ (if the both bits are suitable, let's choose 0). After that we repeat this procedure with a string $x_k b$ and so on. We stop when the condition 3 holds for the interval D_k (the set of the extensions of the latest string). It always happens, because the measure is non-atomic (so $\mu[x_k b_1 \dots b_m]$ tends to 0), and $\mu[x_k b_1 \dots b_{m-1}] < 3\mu[x_k b_1 \dots b_m]$.

Consider a function

$$g(\omega) = \sum_k \frac{\mathbb{I}_{D_k}(\omega)}{2\mu D_k}$$

It is lower semicomputable. To prove that it is probability bounded it is sufficient to show that

$$\mu D_j \geq \sum_{i:\mu D_i < \mu D_j} \mu D_i$$

Indeed, consider the set $\{g(\omega) > C\}$:

$$\mu\{g(\omega) > C\} = \sum_{i:\mu D_i < \frac{1}{2C}} \mu D_i \leq 2 \max\{\mu D_i : \mu D_i < \frac{1}{2C}\} < \frac{1}{C}$$

Consider the ordering π of D_k and denote the elements of this ordering by $\mathbf{D}_k$. The sequence $\mu \mathbf{C}_j^{S_j}$ is exponentially decreasing:

$$\frac{\mu \mathbf{C}_j^{S_j}}{\mu \mathbf{C}_{j+1}^{S_{j+1}}} \geq \mu \mathbf{C}_{j+1}^{S_j - S_{j+1}} = \mu \mathbf{C}_{j+1}^{-z_{j+1}} = 2^{-z_{j+1} \log \mathbf{C}_{j+1}} = 8$$

This inequality shows that $\mathbf{D}_j \subset \mathbf{C}_j$ (because $\mu \mathbf{D}_j > \frac{8}{3}\mu \mathbf{C}_i^{S_i}$ if $i > j$) and moreover

$$\sum_{i:\mu D_i < \mu D_j} \mu D_i = \sum_{l > \pi(j)} \mu \mathbf{D}_l \leq \sum_{k \geq 1} (\frac{8}{3})^{-k} \mu \mathbf{D}_{\pi(j)} < \mu D_j$$

Therefore the function g is probability bounded.

Assume that there exists a constant c such that for all ω

$$\mathbf{t}_\mu^A(\omega) \geq 2^{-c} \frac{g}{\log g}(\omega)$$

Where $\mathbf{t}_\mu^A = 2^{\mathbf{d}_\mu^A(\omega)}$. If $\omega \in D_k$, then for this k there exists a prefix-free set of strings w_l^k such that $\cup_l [w_l^k] \supset D_k$ and

$$\frac{\mathbf{a}(w_l^k)}{\mu([w_l^k])} \geq 2^{-c} \frac{1}{2\mu D_k \log \frac{1}{\mu D_k}}$$

Using the property 3 for large enough k we get:

$$\mu([w_l^k]) \leq 2^{c+1} \mathbf{a}(w_l^k) \mu D_k \log \frac{1}{\mu D_k} < \sqrt{\mu D_k} < \mu C_k$$

Therefore $w_l^k \subset C_k$ and the set $\{w_l^k\}_{k,l}$ is prefix-free.

Consider the following chain of inequalities:

$$\begin{aligned} 1 \geq \sum_{k,l} \mathbf{a}(w_l^k) \geq \sum_{k,l} \mu([w_l^k]) 2^{-c-1} \frac{1}{\mu D_k \log \frac{1}{\mu D_k}} \geq^* \\ \geq^* \sum_k \mu D_k \frac{1}{\mu D_k \log \frac{1}{\mu D_k}} = \sum_k \frac{1}{\log \frac{1}{\mu D_k}} = \\ = \sum_k \frac{1}{\log \frac{1}{\mu \mathbf{D}_k}} =^* \sum_k \frac{1}{S_k \log \frac{1}{\mu \mathbf{C}_k}} \end{aligned}$$

In the proof of lemma 3 we showed that if the series $\sum_n z_n$ does not converge, then the series $\frac{z_n}{S_n}$ where $S_n = \sum_{k \leq n} z_k$ does not converge either, so the right part of the chain of inequalities is ∞. $\square$

Now we can rewrite the chain of inequalities 2 as follows:

$$\mathbf{d}_\mu^E \leq^+ \mathbf{d}_\mu^{\liminf A} \overset{\text{a.e.}}{=\!=} \mathbf{d}_\mu^{\limsup A} \ll \mathbf{d}_\mu^A \ll \mathbf{d}_\mu^P \leq^+ \mathbf{d}_\mu^E + (1+\varepsilon) \log \mathbf{d}_\mu^E$$

where the symbol $\ll$ means that the difference may be greater than $\log \mathbf{d}_\mu$.

One can ask a natural question about the difference between integrable (in the exponential scale) deficiencies $\mathbf{d}_\mu^E$ and $\mathbf{d}_\mu^{\liminf A}$ (or $\mathbf{d}_\mu^{\limsup A}$). We don't know the answer.

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