The 2-Machine Routing Open Shop on a Triangular Transportation Network

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Abstract. The two machine routing open shop being a generalization of the metric TSP and two machine open shop scheduling problem is considered. It is known to be NP-hard even for the simplest case when the transportation network consists of two nodes only. For that simplest case it is known that the optimal makespan for any instance belongs to interval $[\bar{R}, \frac{6}{5}\bar{R}]$, there $\bar{R}$ is the standard lower bound. We generalize that classic result to the case of three-nodes transportation network and present a linear time $\frac{6}{5}$-approximation algorithm for that case.

Keywords: scheduling, routing open shop, optima localization

Introduction

In classic open shop model there are given sets of jobs $\mathcal{J}$ and machines $\mathcal{M}$ and machines have to perform operations of each job in arbitrary order to minimize finish time [6]. The input consists of given processing times for each operation and can be described as $m \times n$ matrix, m and n being the numbers of machines and jobs respectively. It is supposed that after performing an operation of some job machine is immediately available for any successive job to process. However in real life environment the latter is not the case. Jobs usually represent some material objects and therefore some time lags between processing of different operations are often unavoidable.

Various ways to model such time lags are known. Detailed review can be found in [3] and references therein.

We consider the following *routing open shop* model. The jobs are supposed to represent some large immovable objects located at the nodes of some transportation network while machines are mobile and have to travel between the locations of jobs to perform their operations.

The routing open shop model was introduced in [1, 2]. It generalizes two classic NP-hard discrete optimization problems: metric traveling salesman problem (TSP) and open shop scheduling problem.The routing open shop problem can be described as following. There is a transportation network represented by an undirected edge-weighted graph. Nodes represent some *locations* and weight of edge represents a distance between corresponding nodes. One of the nodes is given to be a *depot*. There is a number of *machines* initially located at the depot and a number of *jobs* distributed among all the nodes, each node contains at least one job. Machines have to travel between nodes with unit speed using shortest routes, processing operations of each job in an open shop environment. After performing all the operations machines have to return to the depot. The *makespan* of a schedule is the time moment of returning of the last machine to the depot after processing all the operations. The goal is to minimize the makespan. Following the traditional three-field notation (see [8] for example) we will denote the routing open shop problem as $RO||R_{\max}$.

The routing open shop with a single machine is equivalent to a metric TSP and therefore is well-known to be NP-hard in strong sense. A single-node routing open shop is just a plain open shop problem and is NP-hard for three and more machines while being polynomially solvable in the two-machine case [6]. The simplest non-trivial case of routing open shop is the two-machine problem on a link ($RO2|link|R_{\max}$). This case is shown to be NP-hard in [1]. A fully polynomial time approximation scheme and a few polynomially solvable subcases for $RO2|link|R_{\max}$ are described in [7].

Problem $RO2|link|R_{\max}$ is thoroughly investigated in [2]. It is shown that the optimal makespan for any instance doesn't exceed $\frac{6}{5}\bar{R}$, $\bar{R}$ stands for the standard lower bound (see Section 1), while reaching $\frac{6}{5}\bar{R}$ for some instances. The approximation algorithm described in [2] produces a schedule with makespan belonging to an interval $[\bar{R}, \frac{6}{5}\bar{R}]$, therefore that algorithm provides an $\frac{6}{5}$-approximation.

There are several approximation algorithms known for the general two-machine routing open shop ($RO2||R_{\max}$). An $\frac{7}{4}$-approximation algorithm is described in [1]. More precise $\frac{13}{8}$-approximation algorithm is given in [3]. Note that the $RO2||R_{\max}$ problem includes a metric TSP as a special case. Since the best known algorithm for the metric TSP is the $\frac{3}{2}$-approximation algorithm due to Christofides [5] and Serdyukov [10] we cannot hope to achieve better than $\frac{3}{2}$-approximation for the $RO2||R_{\max}$ problem until a better approximation for the metric TSP will be found. From the other hand the *easy-TSP* version of the $RO2||R_{\max}$ (the case when an optimal solution for the underlying TSP is known or the time complexity of its search is not taken into account) problem admits a $\frac{4}{3}$-approximation algorithm described in [3].

Note that all the approximation algorithms mentioned in the previous paragraph use the standard lower bound $\bar{R}$ to justify their performance guarantees: ρ-approximation algorithm actually obtains a schedule with makespan belonging to an interval $[\bar{R}, \rho\bar{R}]$. Therefore for any instance of the $RO2||R_{\max}$ problem its optimal makespan belongs to the interval $[\bar{R}, \frac{4}{3}\bar{R}]$, though for the case on a link this optima localization interval can be shrinked down to $[\bar{R}, \frac{6}{5}\bar{R}]$. That observations leads us to the question: what is the tightest interval of form $[\bar{R}, \rho\bar{R}]$ which contains optima for all the instances of the $RO2||R_{\max}$ problem? From the previous research we know that $\frac{6}{5} \leqslant \rho \leqslant \frac{4}{3}$.

This paper addresses that question for a case of triangular transportation network. For that $RO2|triangle|R_{\max}$ problem we show that optimum of any instance belongs to the interval $[\bar{R}, \frac{6}{5}\bar{R}]$ hence generalizing the known result for a link [2].

Previously the routing open shop on a triangular transportation network was addressed in [4] for the preemptive version of the problem. It was shown that for any instance of the $RO2|triangle, pmtn|R_{\max}$ problem its optimum belongs to inteval $[\bar{R}, \frac{11}{10}\bar{R}]$ while the algorithmic complexity of the problem is still unknown. As for the $RO2|link, pmtn|R_{\max}$ problem it is shown in [9] that the problem is polynomially solvable and optimum always coincides with the standard lower bound $\bar{R}$.

The structure of the paper is the following. Section 1 contains the formal description of the problem under consideration, necessary notation and preliminary results. In Section 2 we will provide the proof of the main result for three important special cases. The final proof and the description of the $\frac{6}{5}$-approximation algorithm for the general $RO2|triangle|R_{\max}$ problem as well as concluding remarks will be given in Section 3.

1 Preliminary Notes

1.1 Formal description and necessary notation

Let us give a formal description of the $RO2||R_{\max}$ problem.

There are given sets $\mathcal{J} = \{J_1, \dots, J_n\}$ of jobs and $\mathcal{M} = \{A, B\}$ of machines. Each job J_j consists of two operations a_j and b_j to be processed by machines A and B respectively in an arbitrary order. An undirected transportation network is represented by a connected edge-weighted graph $G = \langle V, E\rangle$, $V = \{v_0, \dots, v_{c-1}\}$. The weight ω_{pq} of edge $e_{pq} = [v_p, v_q] \in E$ represents distance between nodes v_p and v_q. Distances are symmetric and satisfy the triangle inequality. Graph G is not necessary complete but we will use the notation ω_{pq} for distance between any two even nonadjacent nodes. Jobs from $\mathcal{J}$ are distributed between the nodes of transportation network and each node contains at least one job. Both machines are initially located at v_0 (the *depot*) and have to travel with unit speed between nodes to perform operations of the jobs. Machines have to return to the depot after completing all the jobs in some arbitrary sequence without preemption.

The goal is to construct a feasible schedule of processing all the operations and returning to the depot in minimal time.

Notation a_j (b_j) will also be used for the processing time of corresponding operation. The set of jobs located at node v_k will be denoted as $\mathcal{J}^k$.

As preemption is not allowed any schedule S can be described by specifying the starting times $s_{jA}(S)$ ($s_{jB}(S)$) for operations a_j (b_j) of each job J_j. Completion time $c_{jA}(S)$ can be defined as $s_{jA}(S) + a_j$, $c_{jB}(S) = s_{jB}(S) + b_j$.

For any feasible schedule S if machine $M \in \mathcal{M}$ processes job $J_j \in \mathcal{J}^k$ before job $J_i \in \mathcal{J}^l$ then the following condition should be carried out:

$$s_{iM}(S) \geqslant c_{jM}(S) + \omega_{kl}.$$

If job $J_j \in \mathcal{J}^k$ is the first job processed by machine M in schedule S then the following condition should hold:

$$s_{jM}(S) \geqslant \omega_{0k}.$$

Let job $J_j \in \mathcal{J}^k$ be the last job processed by machine $M \in \mathcal{M}$ in schedule S. Then machine M releases from duty at time moment

$$R_M(S) \doteq c_{jM}(S) + \omega_{k0}.$$

The makespan of a schedule S is defined as $R_{\max}(S) \doteq \max\{R_A(S), R_B(S)\}$.

We will omit the notation of schedule S in cases it is uniquely defined by the context.

For any problem instance I with weight function ω we will use the following additional notation:

- $\ell_A(I) = \sum\limits_{j=1}^{n} a_j$, $\ell_B(I) = \sum\limits_{j=1}^{n} b_j$ are the *loads of machines* A and B correspondingly;
- $d_j(I) = a_j + b_j$ is the *length of job* $J_j \in \mathcal{J}$;
- $\ell_{\max}(I) = \max\{\ell_A(I), \ell_B(I)\}$ is the maximum machine load;
- $d^k_{\max}(I) = \max\limits_{J_j \in \mathcal{J}^k} d_j(I)$ is the maximum job length at node v_k;
- T^* is the length of the minimal travel route;
- $R^*_{\max}(I)$ stands for the optimal makespan.

Now we can describe the standard lower bound for the optimal makespan introduced in [1]:

$$R^*_{\max}(I) \geqslant \bar{R}(I) \doteq \max\left\{\ell_{\max}(I) + T^*, \max_k \left(d^k_{\max}(I) + 2\omega_{0k}\right)\right\}. \qquad (1)$$

We will focus on a special case $RO2|triangle|R_{\max}$ in which graph G is triangular and $V = \{v_0, v_1, v_2\}$. Lets introduce notation specific to the triangular case:

- $\tau \doteq \omega_{01}$, $\nu \doteq \omega_{12}$, $\mu \doteq \omega_{02}$;
- $T^* = \tau + \mu + \nu$.

In this case the standard lower bound has the following simplified form:

$$\bar{R} = \max\left\{\ell_{\max} + T^*, d^0_{\max}, d^1_{\max} + 2\tau, d^2_{\max} + 2\mu\right\}. \qquad (2)$$

1.2 Jobs' aggregation

We will use the following definition introduced in [4].

Definition 1. *The* load of node v_k *is the total processing time of all operations from that node:*

$$\Delta^k \doteq \sum_{J_j \in \mathcal{J}^k} d_j.$$

Node v_k is referred to as overloaded *if*

$$\Delta^k + 2\omega_{0k} > \bar{R},$$

otherwise the node is underloaded*.*

The following statement holds for any instance of $RO2||R_{\max}$.

Statement 1. *Let I be an instance of $RO2||R_{\max}$ with graph $G = \langle V, E\rangle$. Then V contains at most one overloaded node.*

Proof. Note that due to (1) the following unequality holds for the *total load*:

$$\Delta \doteq \sum_{k=0}^{c-1} \Delta^k = \ell_A + \ell_B \leqslant 2(\bar{R} - T^*). \tag{3}$$

Suppose we have an overloaded node v_k, $\Delta^k > \bar{R} - 2\omega_{0k} \geqslant \bar{R} - T^*$. Then for any other node v_l unequality (3) implies

$$\Delta^l \leqslant \Delta - \Delta^k < 2(\bar{R} - T^*) - \bar{R} + T^* = \bar{R} - T^*,$$

therefore

$$\Delta^l + 2\omega_{0l} \leqslant \Delta^l + T^* < \bar{R}. \ \square$$

The algorithm we'll present is based on the following operation of jobs' aggregation.

Definition 2. *Let I be some instance of $RO2||R_{\max}$, $\mathcal{K} \subseteq \mathcal{J}^k$ for some v_k. Then we say that instance I' is obtained from I by* aggregation *of jobs from $\mathcal{K}$ if*

$$\mathcal{J}^k(I') = \mathcal{J}^k(I) \setminus \mathcal{K} \cup \{J_{\mathcal{K}}\}, \ \ a_{\mathcal{K}} = \sum_{J_j \in \mathcal{K}} a_j, \, b_{\mathcal{K}} = \sum_{J_j \in \mathcal{K}} b_j,$$

$$\forall l \neq k \ \mathcal{J}^l(I') = \mathcal{J}^l(I).$$

The instance $\tilde{I}$ obtains from I by a series of job's aggregation will be referred to as a modification *of I.*

It is obvious that any feasible schedule for some modification $\tilde{I}$ of I can be treated as a feasible schedule for I with the same makespan, therefore the optimum of any modification of I is greater or equal to $R^*_{\max}(I)$. Note that machine loads and node loads are preserved by any job's aggregation operation.

Statement 2. *For any instance I of $RO2||R_{\max}$ there exists its modification $\tilde{I}$ such that*

1. $\bar{R}(\tilde{I}) = \bar{R}(I)$,
2. every underloaded node in $\tilde{I}$ *contains exactly one job, the only overloaded node (if any) contains at most three jobs.*

Proof. In order to preserve the standard lower bound (1) under the job's aggregation operation we may only choose such sets $\mathcal{K} \subseteq \mathcal{J}^k$ that

$$\sum_{J_j \in \mathcal{K}} d_j \leqslant \bar{R} - 2\omega_{0k}.$$

Therefore for any underloaded node v_k aggregation of jobs from set $\mathcal{J}^k$ doesn't increase the standard lower bound. From Statement 1 there is at most one overloaded node v_l in I. Let's prove that we can aggregate jobs from $\mathcal{J}^l$ into at most three jobs preserving $\bar{R}$.

Let $\mathcal{J}^l = \{J_1, \dots, J_p\}$. Let j be the maximal index such that

$$\sum_{t=1}^{j} d_t \leqslant \bar{R} - 2\omega_{0l}.$$

Note that $j < p$ because v_l is overloaded. Perform the aggregation operation for the set $\mathcal{K} = \{J_1, \dots, J_j\}$. Due to the choice of index j we have $d_{\mathcal{K}} + d_{j+1} > \bar{R} - 2\omega_{0l}$. Suppose $j + 1 < p$ (otherwise we have two jobs at v_l and statement is correct). Let $\mathcal{K}' = \{J_{j+2}, \dots, J_p\}$. From (3) we have

$$\sum_{J_t \in \mathcal{K}'} d_j \leqslant \Delta - d_{\mathcal{K}} - d_{j+1} < 2(\bar{R} - T^*) - (\bar{R} - 2\omega_{0l}) \leqslant \bar{R} - T^* \leqslant \bar{R} - 2\omega_{0l},$$

therefore aggregation of set $\mathcal{K}'$ doesn't increase $\bar{R}$. Thus the modification claimed to exist is achieved by aggregation operations of all jobs at each underloaded node, of set $\mathcal{K}$ and of set $\mathcal{K}'$. □

Note that for any instance I the modification $\tilde{I}$ described can be found in $O(n)$ time.

Let $\tilde{I}$ be a modification of I, $\bar{R}(\tilde{I}) = \bar{R}(I) = \bar{R}$. If there exists a schedule S for $\tilde{I}$ such that $R_{\max}(S) \leqslant \rho\bar{R}$ then $R^*_{\max}(I) \leqslant \rho\bar{R}$. Hence it is sufficient to establish the optima localization interval for an instance with small number of jobs which exists due to Statement 2. Such results are described in Section 2.

2 Three important subcases

We will use the branch-and-bounds method to prove the main result for special subcases with small number of jobs. Analogous approach was described in [11], and also used for obtaining similar optima localization results in [2, 4, 9]. The underlying idea is to describe a subset of instances by the choice of critical path in a digraph which represents a partial order of operations used to build an early schedule. Such digraphs determine an order of operations for each job and each machine and will be referred to as *schemes* of a schedule, weights of vertices are correspondent operations' processing times and weights of arcs are travel distances. By $S_{\mathcal{H}}$ we will denote the early schedule built according to the scheme $\mathcal{H}$ for some current instance.

Depending on actual instance the makespan of schedule $S_{\mathcal{H}}$ can be described by different complete paths in $\mathcal{H}$. Knowing which path is critical (i.e. has the maximal length) we can describe the makespan of a $S_{\mathcal{H}}$ by that path's length, i.e. by a sum of weights of nodes and arcs of that path. The enumeration of such complete (and potentially critical) paths lies underneath the branching procedure of the proof.

2.1 All nodes are underloaded

Lemma 1. *Let I be an instance of the $RO2|triangle|R_{\max}$ problem with single job at each node. Then there exists a feasible schedule S for I such that $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.*

Proof. Let us have set of jobs $\mathcal{J}^0 = \{J_0\}$, $\mathcal{J}^1 = \{J_\alpha\}$ and $\mathcal{J}^2 = \{J_\beta\}$. Without lost of generality assume that

$$a_\alpha \geqslant b_\beta. \tag{4}$$

If that is not the case we can renumerate nodes and/or machines to achieve the condition above.

Now we will consider a series of schedules and prove that at least one of them satisfies the lemma's claim.

Consider the schedule $S_1 = S_{\mathcal{H}_1}$ (see Fig. 1). S and F mark the start and finish time moments respectively.

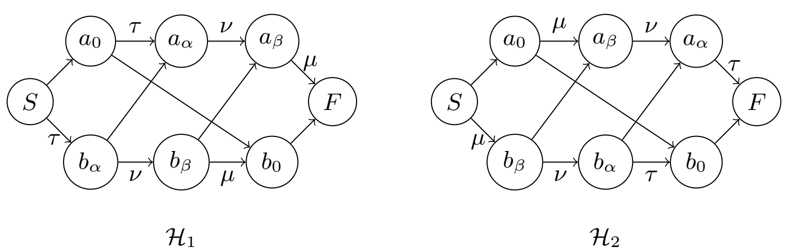

Fig. 1. Schemes $\mathcal{H}_1$ and $\mathcal{H}_2$.

Following a well-known fact from project scheduling the makespan of S_1 coincides with the weighted length of a critical path of $\mathcal{H}_1$. Therefore

$$\begin{aligned}R_{\max}(S_1) = \max\{ & a_0 + \tau + a_\alpha + \nu + a_\beta + \mu, \tau + b_\alpha + \nu + b_\beta + \mu + b_0, a_0 + b_0,\\ & \tau + b_\alpha + a_\alpha + \nu + a_\beta + \mu, \tau + b_\alpha + \nu + b_\beta + a_\beta + \mu\}.\end{aligned}$$

From (2) the first three sums from the max clause above clearly don't exceed the lower bound $\bar{R}$. If one of the correspondent paths turns out to be critical then $R_{\max}(S_1) = \bar{R}$ and the claim of the lemma follows immediately. Further such to be called *trivial* paths will be excluded from consideration.

Using assumption (4) we can conclude that

$$R_{\max}(S_1) = \tau + b_\alpha + a_\alpha + \nu + a_\beta + \mu = T^* + b_\alpha + a_\alpha + a_\beta. \tag{5}$$

Now let $S_2 = S_{\mathcal{H}_2}$ (Fig. 1). Using similar reasoning and excluding trivial paths we conclude that

$$R_{\max}(S_2) = \max\{T^*+b_\beta+a_\beta+a_\alpha, T^*+b_\beta+b_\alpha+a_\alpha\} = T^*+b_\beta+a_\alpha+\max\{a_\beta, b_\alpha\}.$$

Note that due to the metric property of the distances the makespan of S_2 can be evaluated as

$$R_{\max}(S_2) \leqslant T^* + a_\alpha + \bar{R} - 2\mu \tag{6}$$

or as

$$R_{\max}(S_2) \leqslant T^* + b_\beta + \bar{R} - 2\tau. \tag{7}$$

Consider schedules $S_3 = S_{\mathcal{H}_3}$ and $S_4 = S_{\mathcal{H}_4}$ (see Fig. 2). There is the only

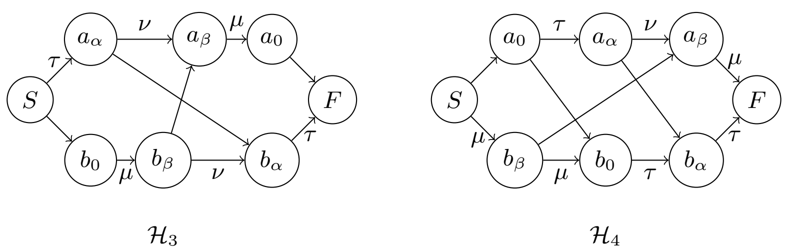

Fig. 2. Schemes $\mathcal{H}_3$ and $\mathcal{H}_4$.

non-trivial path in $\mathcal{H}_3$ therefore

$$R_{\max}(S_3) = b_0 + b_\beta + a_\beta + a_0 + 2\mu. \tag{8}$$

The scheme $\mathcal{H}_4$ contains three non-trivial paths:

1. $S \to a_0 \xrightarrow{\tau} a_\alpha \to b_\alpha \xrightarrow{\tau} F$;
2. $S \to a_0 \to b_0 \xrightarrow{\tau} b_\alpha \xrightarrow{\tau} F$;
3. $S \xrightarrow{\mu} b_\beta \xrightarrow{\mu} b_0 \xrightarrow{\tau} b_\alpha \xrightarrow{\tau} F$.

We will consider those cases one by one.

Case 1:

$$R_{\max}(S_4) = a_0 + a_\alpha + b_\alpha + 2\tau. \tag{9}$$

Let S be the best schedule among $S_1, \ldots, S_4$. Using (2), (5), (6), (8) and (9) we obtain

$$5R_{\max}(S) \leqslant R_{\max}(S_1) + R_{\max}(S_2) + 2R_{\max}(S_3) + R_{\max}(S_4) \leqslant$$
$$\leqslant (T^* + b_\alpha + a_\alpha + a_\beta) + (T^* + a_\alpha + \bar{R} - 2\mu) + 2(b_0 + b_\beta + a_\beta + a_0 + 2\mu) +$$
$$+ (a_0 + a_\alpha + b_\alpha + 2\tau) = \bar{R} + 2T^* + 2\mu + 2\tau + 3\ell_A + 2\ell_B \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 2:

$$R_{\max}(S_4) = a_0 + b_0 + b_\alpha + 2\tau. \tag{10}$$

Again, let S be the best schedule among $S_1, \ldots, S_4$. Using (2), (5), (6), (7), (8) and (10) we obtain

$$5R_{\max}(S) \leqslant R_{\max}(S_1) + 2R_{\max}(S_2) + R_{\max}(S_3) + R_{\max}(S_4) \leqslant$$
$$\leqslant (T^* + b_\alpha + a_\alpha + a_\beta) + (T^* + b_\beta + \bar{R} - 2\tau) + (T^* + a_\alpha + \bar{R} - 2\mu) +$$
$$+ (b_0 + b_\beta + a_\beta + a_0 + 2\mu) + (a_0 + b_0 + b_\alpha + 2\tau) = 2\bar{R} + 3T^* + 2\ell_A + 2\ell_B \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 3:

$$R_{\max}(S_4) = 2\mu + 2\tau + \ell_B. \tag{11}$$

In this case we consider one more schedule $S_5 = S_{\mathcal{H}_5}$ (see Fig. 3).

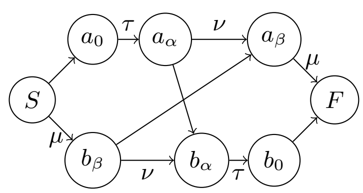

Fig. 3. Scheme $\mathcal{H}_5$.

There is the only non-trivial path in $\mathcal{H}_5$ therefore we may assume

$$R_{\max}(S_5) = a_0 + b_0 + 2\tau + a_\alpha + b_\alpha. \tag{12}$$

Let S be the best schedule among $S_1, \ldots, S_5$. Using (2), (5), (7), (8), (11) and (12) we obtain

$$5R_{\max}(S) \leqslant R_{\max}(S_1) + R_{\max}(S_2) + R_{\max}(S_3) + R_{\max}(S_4) + R_{\max}(S_5) \leqslant$$
$$\leqslant (T^* + b_\alpha + a_\alpha + a_\beta) + (T^* + b_\beta + \bar{R} - 2\tau) + (b_0 + b_\beta + a_\beta + a_0 + 2\mu) +$$
$$+ (2\mu + 2\tau + \ell_B) + (a_0 + b_0 + 2\tau + a_\alpha + b_\alpha) \leqslant \bar{R} + 5T^* + 2\ell_A + 3\ell_B \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

This concludes the proof of Lemma 1. $\square$

2.2 The depot is overloaded

Lemma 2. *Let I be an instance of the $RO2|triangle|R_{\max}$ problem with single job at each node except the depot which is overloaded and contains at most three jobs. Then there exists a feasible schedule S for I such that $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.*

Proof. Let us use the following notation for sets of jobs: $\mathcal{J}^0 = \{J_1, J_2, J_3\}$, $\mathcal{J}^1 = \{J_\alpha\}$ and $\mathcal{J}^2 = \{J_\beta\}$. If the depot contains only two jobs we will add dummy job J_3 with zero processing times.

Without lost of generality we can assume that

$$a_2 \geqslant b_1,\ a_3 \geqslant b_2. \tag{13}$$

Indeed, we can always achieve that condition by proper re-numeration of mahcines or/and jobs from $\mathcal{J}^0$ due to the following reasoning. Consider three pairs of operations: a_2 and b_1, a_3 and b_2, a_1 and b_3, and compare them pairwise. Without lost of generality due to possible re-numeration of machines at least for two of those pairs operation of machine A is greater or equal to the respective operation of machine B. Using proper numeration of jobs we can assure that (13) holds.

Note that as the depot v_0 is overloaded we have $d_1 + d_2 + d_3 > \bar{R}$. Since $\sum\limits_j d_j = \ell_A + \ell_B \leqslant 2\bar{R} - 2T^*$ we have

$$d_\alpha + d_\beta + 2T^* < 2(\bar{R} - T^*) - \bar{R} + 2T^* \leqslant \bar{R}. \tag{14}$$

Now consider a schedule $S_1 = S_{\mathcal{H}_1}$ (see Fig. 4). Note that complete paths containing dotted arcs cannot be critical due to the assumption (13). We will not consider such paths and omit reference to that assumption further. Also the length of the path containing dashed arc is at most $\bar{R}$ due to (14). Therefore we have to consider just one non-trivial path $S \to a_1 \to a_2 \to a_3 \to b_3 \to F$:

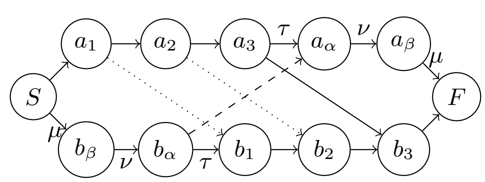

Fig. 4. Scheme $\mathcal{H}_1$.

$$R_{\max}(S_1) = a_1 + a_2 + a_3 + b_3. \tag{15}$$

Consider a schedule $S_2 = S_{\mathcal{H}_2}$ (see Fig. 5). We need to consider just one path $S \xrightarrow{\mu} b_\beta \xrightarrow{\nu} b_\alpha \xrightarrow{\tau} b_3 \to a_3 \xrightarrow{\tau} a_\alpha \xrightarrow{\nu} a_\beta \xrightarrow{\mu} F$:

$$R_{\max}(S_2) = 2T^* + b_\beta + b_\alpha + b_3 + a_3 + a_\alpha + a_\beta. \tag{16}$$

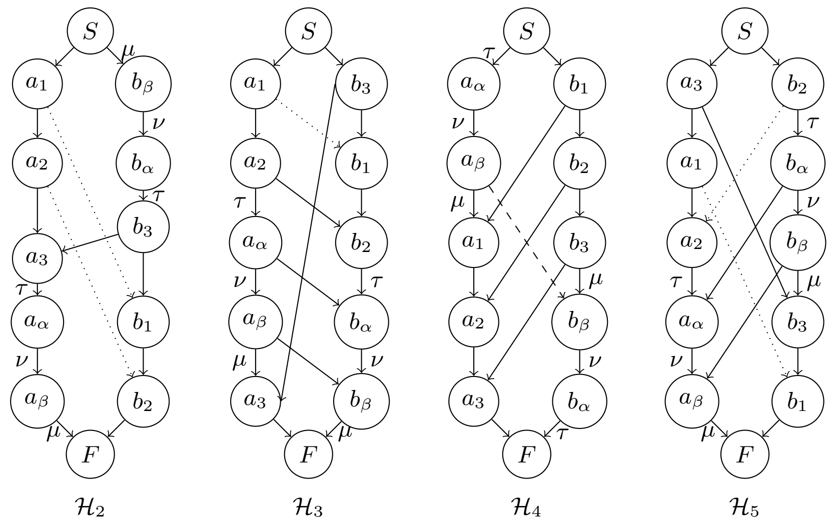

Fig. 5. Schemes $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_4$ and $\mathcal{H}_5$.

Now consider schedules $S_3 = S_{\mathcal{H}_3}$ and $S_4 = S_{\mathcal{H}_4}$ (Fig. 5). We need to consider three non-trivial paths in $\mathcal{H}_3$ and due to (14) only three paths in $\mathcal{H}_4$.

Case 1:
$$R_{\max}(S_3) = T^* + a_1 + a_2 + a_\alpha + b_\beta + \max\{a_\beta, b_\alpha\}. \tag{17}$$

Case 1.1:
$$R_{\max}(S_4) = b_1 + a_2 + a_3 + \max\{a_1, b_2\}. \tag{18}$$

Here and further let S be the best schedule among all schedules built in each case. Then from (15), (16), (17) and (18) we have

$$5R_{\max}(S) \leqslant R_{\max}(S_1)+R_{\max}(S_2)+R_{\max}(S_3)+2R_{\max}(S_4) \leqslant 4\ell_A+2\ell_B+3T^* \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 1.2:
$$R_{\max}(S_4) = b_1 + b_2 + b_3 + a_3. \tag{19}$$
From (17) and (19) we have

$$2R_{\max}(S) \leqslant R_{\max}(S_3) + R_{\max}(S_4) \leqslant \ell_A + \ell_B + T^* \leqslant 2\bar{R},$$

therefore $R_{\max}(S) = \bar{R}$.

Case 2:
$$R_{\max}(S_3) = T^* + a_1 + a_2 + b_2 + b_\alpha + b_\beta. \tag{20}$$

Case 2.1:
$$R_{\max}(S_4) = b_1 + a_1 + a_2 + a_3. \tag{21}$$
In this case from (15), (16), (20) and (21) we have

$$5R_{\max}(S) \leqslant R_{\max}(S_1)+R_{\max}(S_2)+R_{\max}(S_3)+2R_{\max}(S_4) \leqslant 4\ell_A+2\ell_B+3T^* \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 2.2:
$$R_{\max}(S_4) = b_1 + b_2 + b_3 + a_3. \tag{22}$$
In this case from (15), (16), (20) and (22) we have

$$5R_{\max}(S) \leqslant R_{\max}(S_1)+R_{\max}(S_2)+2R_{\max}(S_3)+R_{\max}(S_4) \leqslant 3\ell_A+3\ell_B+3T^* \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 2.3:
$$R_{\max}(S_4) = b_1 + b_2 + a_2 + a_3. \tag{23}$$
In this case we build one last schedule $S_5 = S_{\mathcal{H}_5}$ (Fig. 5). Consider three non-trivial paths in $\mathcal{H}_5$.

Case 2.3.1:
$$R_{\max}(S_5) = a_3 + b_3 + b_1. \tag{24}$$
In this case from (20) and (24) we have

$$2R_{\max}(S) \leqslant R_{\max}(S_3) + R_{\max}(S_5) \leqslant \ell_A + \ell_B + 2T^* \leqslant 2\bar{R},$$

therefore $R_{\max}(S) = \bar{R}$.

Case 2.3.2:
$$R_{\max}(S_5) = b_2 + b_\alpha + a_\beta + \max\{a_\alpha, b_\beta\} + T^*. \tag{25}$$
In this case from (15) and (25) we have

$$2R_{\max}(S) \leqslant R_{\max}(S_1) + R_{\max}(S_5) \leqslant \ell_A + \ell_B + 2T^* \leqslant 2\bar{R},$$

therefore $R_{\max}(S) = \bar{R}$.

This concludes the proof of Lemma 2. □

2.3 Some distant node is overloaded

Lemma 3. *Let I be an instance of the $RO2|triangle|R_{\max}$ problem with single job at each node except one of distant nodes which is overloaded and contains at most three jobs. Then there exists a feasible schedule S for I such that $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.*

The proof of Lemma 3 can be found in Appendix A.

3 Conclusion

Statement 2, Lemma 1, Lemma 2 and Lemma 3 imply the following

Theorem 1. *For any instance I of $RO2|triangle|R_{\max}$ there exists a feasible schedule S with makespan from interval $[\bar{R}, \frac{6}{5}\bar{R}]$. Such schedule can be found in linear time.*

Indeed, we just need to perform the jobs' aggregation procedure described in proof of Statement 2 to obtain modification $\tilde{I}$, then use the proof of correspondent Lemma according to the existance and location of overloaded node to build a feasible schedule for $\tilde{I}$ with makespan from $[\bar{R}, \frac{6}{5}\bar{R}]$, and finally transform that schedule into the feasible schedule for initial instance, treating each aggregated operation as a block of initial operations performed without idles in arbitrary order.

Note that the interval $[\bar{R}, \frac{6}{5}\bar{R}]$ is a tight optima localization interval for $RO2|triangle|R_{\max}$ problem as it is for its special case $RO2|link|R_{\max}$ [2].

The most important question still to be investigated is the following

Open Question 1. *What is the smallest value ρ such that interval $[\bar{R}, \rho\bar{R}]$ is an optima localization interval for $RO2||R_{\max}$ problem?*

The second interesting question concerns the possibility of generalizing results for $RO2|link|R_{\max}$ from [7] to our case $RO2|triangle|R_{\max}$. Those results (polynomially solvable subcases and an FPTAS) are based on the properties of the Gonzalez-Sáhni algorithm for two-machine open shop problem [6].

The Gonzalez-Sáhni algorithm consists of three main steps.

Step 1. Separate all jobs from $\mathcal{J}$ into two subsets:

$$\mathcal{J}_{\leqslant} \doteq \{J_j \in \mathcal{J} | a_j \leqslant b_j\} \text{ and } \mathcal{J}_{>} \doteq \{J_j \in \mathcal{J} | a_j > b_j\}.$$

Step 2. Choose the *diagonal* job J_r such that the maximum

$$\max\{\max\{a_j | J_j \in \mathcal{J}_{\leqslant}\}, \max\{b_j | J_j \in \mathcal{J}_{>}\}$$

is reached at J_r. Without lost of generality $J_r \in \mathcal{J}_{\leqslant}$.

Step 3. Sequence operations of machine A in an arbitrary order such that operations of jobs from $\mathcal{J}_{\leqslant} \setminus \{J_r\}$ precede operations of jobs from $\mathcal{J}_{>}$ and a_r is the last operation processed by A. Operations of machine B are sequenced in the same order except for b_r which is processed first.

Also note that if $d_r \geqslant \ell_{\max}$ then sequence for machine A at Step 3 can be arbitrary providing that a_r is the last operation of A.

Two theorems from [7] state the following:
For any instance I of $RO2|link|R_{\max}$

1. *if $J_r \in \mathcal{J}^0$ then optimum of I equals to $\bar{R}$ and can be found in linear time;*
2. *if $J_r \in \mathcal{J}^1$ and $d_r \geqslant \ell_{\max}$ then optimum of I equals to $\bar{R}$ and can be found in linear time;*
3. *otherwise a schedule for I of makespan $\ell_{\max}+2T^*$ can be built in linear time.*

Similar technique can be used to prove the following

Lemma 4. *For any instance I of $RO2|triangle|R_{\max}$*

1. *if $J_r \in \mathcal{J}^0$ then a schedule for I of makespan $\ell_{\max} + 2(\tau + \nu)$ can be built in linear time;*

2. *if $J_r \in \mathcal{J}^1$ and $d_r \geqslant \ell_{\max}$ then optimum of I equals to $\bar{R}$ and can be found in linear time;*
3. *otherwise a schedule for I of makespan $\ell_{\max}+2T^*$ can be built in linear time.*

As we see the first case ($J_r \in \mathcal{J}^0$) resolves differently for those two problems. Therefore the technique from [7] will not help us to get an FPTAS for $RO2|triangle|R_{\max}$ that easily. This observation leads us to the following

Open Question 2. *Does an FPTAS for $RO2|triangle|R_{\max}$ exist?*

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A Proof of Lemma 3

Proof. Without lost of generality let v_1 be the overloaded node. Let us have sets of jobs $\mathcal{J}^0 = \{J_0\}$, $\mathcal{J}^1 = \{J_1, J_2, J_3\}$ and $\mathcal{J}^2 = \{J_\beta\}$. Without lost of generality similar to the proof of Lemma 2 we can assume that

$$a_2 \geqslant b_1,\ a_3 \geqslant b_2. \tag{26}$$

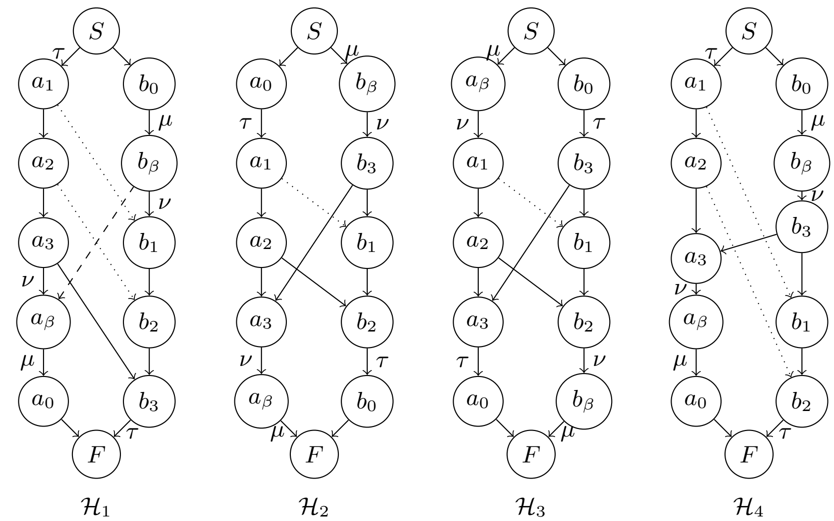

Fig. 6. Schemes $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$.

Note that as node v_1 is overloaded we have $d_1 + d_2 + d_3 > \bar{R} - 2\tau$. Since $\sum_j d_j = \ell_A + \ell_B \leqslant 2\bar{R} - 2T^*$ we have

$$d_0 + d_\beta + 2\mu + 2\nu < 2(\bar{R} - T^*) - (\bar{R} - 2\tau) + 2\mu + 2\nu \leqslant \bar{R}. \tag{27}$$

Let $S_1 = S_{\mathcal{H}_1}$, Fig. 6. Note that paths containing dotted arcs cannot be critical due to the assumption (26). We will omit reference to that assumption further.

Using (27) we can exclude the path containing the dashed arc and therefore we have to consider only $S \xrightarrow{\tau} a_1 \to a_2 \to a_3 \to b_3 \xrightarrow{\tau} F$:

$$R_{\max}(S_1) = 2\tau + a_1 + a_2 + a_3 + b_3. \tag{28}$$

Now consider schedule $S_2 = S_{\mathcal{H}_2}$ (see Fig. 6). We only have to consider two non-trivial paths in $\mathcal{H}_2$.

Case 1:
$$R_{\max}(S_2) = 2\tau + a_0 + a_1 + a_2 + b_2 + b_0. \tag{29}$$

Consider the next schedule $S_3 = S_{\mathcal{H}_3}$ (Fig. 6). Again, we have to consider two possibilities for the makespan of schedule S_3.

Case 1.1:
$$R_{\max}(S_3) = a_\beta + a_1 + a_2 + b_2 + b_\beta + 2\nu + 2\mu. \tag{30}$$

Let $S_4 = S_{\mathcal{H}_4}$ (Fig. 6). We have to consider just one non-trivial complete path $S \to b_0 \xrightarrow{\mu} b_\beta \xrightarrow{\nu} b_3 \to a_3 \xrightarrow{\nu} a_\beta \xrightarrow{\mu} a_0 \to F$:

$$R_{\max}(S_4) = b_0 + b_\beta + b_3 + a_3 + a_\beta + a_0 + 2\mu + 2\nu. \tag{31}$$

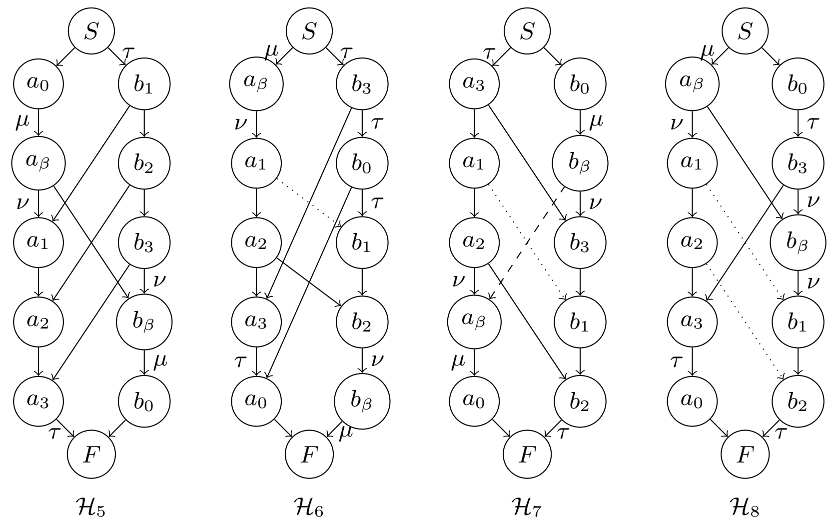

Fig. 7. Schemes $\mathcal{H}_5$, $\mathcal{H}_6$, $\mathcal{H}_7$ and $\mathcal{H}_8$.

Here and further S will denote the best among all schedules built in each case. Using (28), (29), (30), (31) and (2) we have

$$5R_{\max}(S) \leqslant R_{\max}(S_1)+R_{\max}(S_2)+R_{\max}(S_3)+2R_{\max}(S_4) \leqslant 6\ell_{\max}+6T^* \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 1.2:
$$R_{\max}(S_3) = b_0 + b_3 + a_3 + a_0 + 2\tau. \tag{32}$$

Consider the schedule $S_4 = S_{\mathcal{H}_5}$ (see Fig. 7). Now we consider all four non-trivial paths in $\mathcal{H}_5$.

Case 1.2.1:
$$R_{\max}(S_4) = a_0 + a_\beta + b_\beta + b_0 + 2\mu. \tag{33}$$
In this case due to (28) and (33) the makespan of S doesn't exceed

$$\frac{1}{2}\left(R_{\max}(S_1) + R_{\max}(S_4)\right) \leqslant \frac{1}{2}(\ell_A + \ell_B + 2T^*) \leqslant \bar{R}.$$

Case 1.2.2:
$$R_{\max}(S_4) = b_1 + a_1 + a_2 + a_3 + 2\tau. \tag{34}$$

Using (28), (29), (32) and (34) we obtain

$$5R_{\max}(S) \leqslant R_{\max}(S_1)+R_{\max}(S_2)+R_{\max}(S_3)+2R_{\max}(S_4) \leqslant 6\ell_{\max}+10\tau \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 1.2.3:
$$R_{\max}(S_4) = b_1 + b_2 + b_3 + a_3 + 2\tau. \tag{35}$$

Using (28), (29), (32) and (35) we obtain

$$5R_{\max}(S) \leqslant R_{\max}(S_1) + 2R_{\max}(S_2) + R_{\max}(S_3) + R_{\max}(S_4) \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 1.2.4:
$$R_{\max}(S_4) = b_1 + b_2 + a_2 + a_3 + 2\tau. \tag{36}$$
We will use one more schedule $S_5 = S_{\mathcal{H}_6}$ (Fig. 7). We have to consider four non-trivial complete paths in $\mathcal{H}_6$.

Case 1.2.4.1:
$$R_{\max}(S_5) = a_\beta + a_1 + a_2 + b_2 + b_\beta + 2\mu + 2\nu. \tag{37}$$
Due to (32) and (37) we have

$$2R_{\max}(S) \leqslant R_{\max}(S_3) + R_{\max}(S_5) = \ell_A + \ell_B + 2T^* \leqslant 2\bar{R},$$

therefore $R_{\max}(S) = \bar{R}$.

Case 1.2.4.2:
$$R_{\max}(S_5) = b_3 + a_3 + a_0 + 2\tau. \tag{38}$$
In this case using (26), (29), (36) and (38) we have

$$5R_{\max}(S) \leqslant 2R_{\max}(S_2) + R_{\max}(S_4) + 2R_{\max}(S_5) \leqslant 4\ell_A + 2\ell_B + 10\tau \leqslant 6\bar{R},$$

thus $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 1.2.4.3:
$$R_{\max}(S_5) = b_3 + b_0 + a_0 + 2\tau. \tag{39}$$
Using (36) and (39) we obtain

$$2R_{\max}(S) \leqslant R_{\max}(S_4) + R_{\max}(S_5) \leqslant \ell_A + \ell_B + 4\tau \leqslant 2\bar{R},$$

therefore $R_{\max}(S) = \bar{R}$.

Case 1.2.4.4:
$$R_{\max}(S_5) = \ell_B + T^* + 2\tau. \tag{40}$$
Using (28), (29), (32), (36) and (40) we obtain

$$5R_{\max}(S) \leqslant \sum_{k=1}^{5} R_{\max}(S_k) \leqslant 3\ell_A + 3\ell_B + T^* + 10\tau \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 1 is considered completely.

Case 2:
$$R_{\max}(S_2) = 2\mu + 2\nu + a_0 + b_\beta + b_3 + a_3 + a_\beta. \tag{41}$$
Consider schedule $S_3 = S_{\mathcal{H}_3}$ (Fig. 6).

Case 2.1:
$$R_{\max}(S_3) = a_\beta + a_1 + a_2 + b_2 + b_\beta + 2\nu + 2\mu. \tag{42}$$
Consider schedule $S_4 = S_{\mathcal{H}_7}$ (Fig. 7). We have to consider two subcases.

Case 2.1.1:
$$R_{\max}(S_4) = a_3 + a_1 + a_2 + b_2 + 2\tau. \tag{43}$$
We will use one more schedule $S_5 = S_{\mathcal{H}_8}$ (Fig. 7).

Case 2.1.1.1:
$$R_{\max}(S_5) = a_\beta + b_\beta + b_1 + b_2 + T^*. \tag{44}$$
Due to (28) and (44) we have

$$2R_{\max}(S) \leqslant R_{\max}(S_1) + R_{\max}(S_5) = \ell_A + \ell_B + 2T^* \leqslant 2\bar{R},$$

therefore $R_{\max}(S) = \bar{R}$.

Case 2.1.1.2:
$$R_{\max}(S_5) = b_0 + b_3 + a_3 + a_0 + 2\tau. \qquad (45)$$
Due to (42) and (45) we have

$$2R_{\max}(S) \leqslant R_{\max}(S_3) + R_{\max}(S_5) = \ell_A + \ell_B + 2T^* \leqslant 2\bar{R},$$

therefore $R_{\max}(S) = \bar{R}$.

Case 2.1.1.3:
$$R_{\max}(S_5) = \ell_B + 2\tau + 2\nu. \qquad (46)$$
Using (28), (41), (42), (43) and (46) we obtain

$$5R_{\max}(S) \leqslant \sum_{k=1}^{5} R_{\max}(S_k) \leqslant 3\ell_A + 3\ell_B + 6T^* \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 2.1.2:
$$R_{\max}(S_4) = a_3 + b_3 + b_1 + b_2 + 2\tau. \qquad (47)$$
Using (28), (41), (42) and (47) we have

$$5R_{\max}(S) \leqslant R_{\max}(S_1) + R_{\max}(S_2) + 2R_{\max}(S_3) + R_{\max}(S_4) \leqslant 3\ell_A + 3\ell_B + 6T^* \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

Case 2.2:
$$R_{\max}(S_3) = b_0 + b_3 + a_3 + a_0 + 2\tau. \qquad (48)$$
We will need one last schedule $S_4 = S_{\mathcal{H}_9}$ (see Fig. 8). There is the only non-trivial

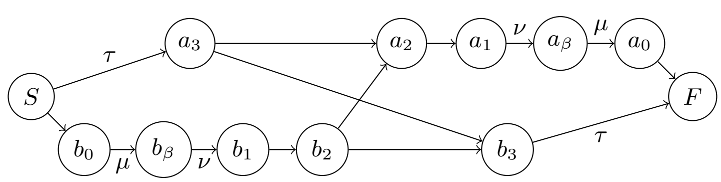

Fig. 8. Scheme $\mathcal{H}_9$.

complete path in that scheme.

$$R_{\max}(S_4) = b_0 + b_\beta + b_1 + b_2 + a_2 + a_1 + a_\beta + a_0 + 2\mu + 2\nu. \qquad (49)$$

Using (28), (41), (48) and (49) we have

$$5R_{\max}(S) \leqslant R_{\max}(S_1) + R_{\max}(S_2) + R_{\max}(S_3) + 4R_{\max}(S_4) \leqslant 3\ell_A + 3\ell_B + 6T^* \leqslant 6\bar{R},$$

therefore $R_{\max}(S) \leqslant \frac{6}{5}\bar{R}$.

This concludes the proof of Lemma 3.

$\square$