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On random partitions induced by random maps

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Abstract

The lattice of the set partitions of $[n]$ ordered by refinement is studied. Given a map $\phi : [n] \to [n]$, by taking preimages of elements we construct a partition of $[n]$. Suppose t partitions $p_1, p_2, \ldots, p_t$ are chosen independently according to the uniform measure on the set of mappings $[n] \to [n]$. The probability that the coarsest refinement of all p_i's is the finest partitions $\{\{1\}, \ldots, \{n\}\}$ is shown to approach 1 for any $t \geq 3$ and $e^{-1/2}$ for $t = 2$. The probability that the finest coarsening of all p_i's is the one-block partition is shown to approach 1 if $t(n) - \log n \to \infty$ and 0 if $t(n) - \log n \to -\infty$. The size of the maximal block of the finest coarsening of all p_i's for a fixed t is also studied.

1 Introduction

For a given n define Π_n to be the set of all partitions of $[n] = \{1, 2, \ldots, n\}$ with partial order given by $p \preceq p'$ if every block of p' is a union of blocks in p. This partially ordered set is known to be a lattice, see [7], that is, for any two partitions $p_1, p_2 \in \Pi_n$ there exists the greatest lower bound $\inf\{p_1, p_2\}$ and the least upper bound $\sup\{p_1, p_2\}$. Namely, $\inf\{p_1, p_2\}$ is the partition given by all the non-empty intersections of blocks of p_1 and p_2, and $\sup\{p_1, p_2\}$ is the smallest partition whose blocks are union of those in both p_1 and p_2.

Every map $\phi : [n] \to [n]$ induces a partition p_ϕ of $[n]$ into non-empty preimages of ϕ: $[n] = \cup_{i:\phi^{-1}(i)\neq\emptyset}\phi^{-1}(i)$. Throughout the paper we work with random partitions of $[n]$ chosen according to the uniform measure on the set of all mappings from $[n]$ to $[n]$.

We study properties of $\inf_{1\leq i\leq t} p_i$ and $\sup_{1\leq i\leq t} p_i$ where p_i are chosen independently. We shall be mostly interested in how likely $\inf_i p_i$ is to be the minimal partition $p_{\min} = \{\{1\}, \ldots, \{n\}\}$ and how likely $\sup_i p_i$ is to be the maximal partition $p_{\max} = \{[n]\}$. Similar questions for the case when partitions are taken according to the uniform measure on the

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set Π_n were studied in great details in [6], see also [1], and for different finite lattices with the uniform measure, see [2, 3].

In order to keep notation more readable we avoid using integer part $\lfloor\cdot\rfloor$ when it is formally needed. So when an argument a is supposed to be integer, say it represents a number of some objects or appears in bounds for summation or product, it should be understood as $\lfloor a\rfloor$.

The rest of the paper is organized as follows. In Section 2 we investigate the infimum of several random partitions; part of these results were claimed by Pittel [6] and we present a proof for the sake of completeness. Section 3 summarizes some known facts about the Stirling numbers of the second kind. Section 4 deals with the supremum of random partitions. In the last section we study the size of the maximal block of $\sup_{1\leq i\leq t} p_i$ for a fixed t.

2 Infimum of several partitions

In this section we study $\inf\{p_{\phi_1},\dots,p_{\phi_t}\}$ where $\phi_1,\dots,\phi_n$ are maps from $[n]$ to $[n]$ chosen independently. The threshold value for t here turns out to be equal to 2: if $t>2$ then the probability that $\inf_i p_{\phi_i}=p_{\min}$ tends to 1 as n tends to infinity, and for $t=2$ this probability tends to $e^{-1/2}$. Evidently, the first fact for $t>3$ would follow from this fact for $t=3$. We now formulate and prove these results.

Theorem 1. *Suppose three maps $\phi_1,\phi_2,\phi_3:[n]\to[n]$ are chosen independently according to the uniform measure on the set of all such maps. Then*

$$\lim_{n\to\infty}\mathbb{P}(\inf\{p_{\phi_1},p_{\phi_2},p_{\phi_3}\}=p_{\min})=1.$$

Proof. We use a simple observation that if $p:=\inf\{p_{\phi_1},p_{\phi_2},p_{\phi_3}\}\neq p_{\min}$ then at least two elements in p must be in the same block. Let A be the set of all pairs $\{i,j\}$ such that i and j are in the same block in p. Then we have

$$\mathbb{P}[\inf\{p_{\phi_1},p_{\phi_2},p_{\phi_3}\}\neq p_{\min}]\leq\mathbb{E}[|A|]=\binom{n}{2}\mathbb{P}[1 \text{ and } 2 \text{ are in the same block in } p].$$

This probability can be easily calculated explicitly and equals $(\mathbb{P}\left[\phi_1(i)=\phi_1(j)\right])^3=n^{-3}$ which gives us

$$\mathbb{P}[\inf\{p_{\phi_1},p_{\phi_2},p_{\phi_3}\}=p_{\min}]=1-\mathbb{P}[\inf\{p_{\phi_1},p_{\phi_2},p_{\phi_3}\}\neq p_{\min}]\geq 1-\binom{n}{2}n^{-3}\to 1.$$

□

The idea of the proof of the next theorem is given in [6], we present it here for the sake of completeness.

Theorem 2. *Suppose two maps $\phi_1,\phi_2:[n]\to[n]$ are chosen independently according to the uniform measure on the set of all such maps. Then*

$$\lim_{n\to\infty}\mathbb{P}[\inf\{p_{\phi_1},p_{\phi_2}\}=p_{\min}]=e^{-1/2}.$$

Proof. Here the argument is a bit more subtle. We denote $\inf\{p_{\phi_1}, p_{\phi_2}\}$ by p. Let A be the set of two-element blocks in p. Let B be the set of triples $\{i, j, k\}$ such that i, j and k are in the same block in p. We first note that

$$\mathbb{E}\left[|B|\right] = \binom{n}{3}\mathbb{P}[1,\ 2 \text{ and } 3 \text{ are in the same block in } p] = \binom{n}{3}n^{-6} < n^{-3}.$$

Hence, with probability at least $1 - n^{-3}$ the partition p has blocks of sizes 1 and 2 only. We now study the random variable $|A|$ which counts the number of two-element blocks in p. In order to evaluate $\mathbb{P}(|A| = 0)$ we first calculate factorial moments of $|A|$. For any fixed $k \geq 0$ we have

$$\begin{aligned}\mathbb{E}\left[\binom{|A|}{k}\right] &= \frac{1}{k!}\prod_{s=0}^{k-1}\binom{n-2s}{2} \cdot \mathbb{P}[\{1,2\}, \dots \{2k-1, 2k\} \text{ are blocks in } p] \\ &= \left(1 + O\left(\frac{1}{n}\right)\right) \cdot \frac{n^{2k}}{2^k \cdot k!} \cdot \frac{1}{n^{2k}} = \frac{1 + o(1)}{2^k \cdot k!}, \qquad n \to \infty,\end{aligned}$$

where $o(1)$ is uniform in k. Now it is easy to see that

$$\mathbb{P}\left[|A| = 0\right] = \sum_{k=0}^{\infty}(-1)^k \cdot \mathbb{E}\left[\binom{|A|}{k}\right] = (1 + o(1)) \cdot \sum_{k=0}^{\infty}\frac{(-1)^k}{2^k \cdot k!} \to e^{-1/2}, \qquad n \to \infty.$$

□

3 Some properties of the Stirling numbers of the second kind

In order to estimate the probability that supremum of several random partitions is equal to $p_{\max}$ we shall need the notion of the Stirling numbers of the second kind. Recall that the Stirling number of the second kind $S(n, k)$ counts the number of ways to partition the set $[n]$ into k blocks. It is clear that the number of surjective maps from $[k]$ to $[l]$ equals $S(k, l) \cdot l!$ as each such map gives rise to a partition of $[k]$ into l blocks. We shall frequently use this fact in our calculations. The following well-known fact and its corollary shall not be used in our arguments though it is useful to keep them in mind.

Lemma 3.1 ([5, Theorem 3.2])**.** *For a given n, the Stirling numbers of the second kind, $S(n, k)$ form a log-concave sequence in k. That is, for any $k = 2 \dots n-1$ we have*

$$S(n, k)^2 \geq S(n, k-1) \cdot S(n, k+1).$$

Corollary 3.1. *For any natural number n the quantity $\frac{S(n,k-1)}{S(n,k)}$ increases in k.*

For the proof of the next lemma concerning the Stirling numbers of the second kind, we need the so-called multi-valued map principle.

Multi-valued map principle. Let f be a multi-valued map from a finite set S to a finite set T. For $t \in T$ write $f^{-1}(t) := \{s \in S : t \in f(s)\}$. Then

$$\frac{|S|}{|T|} \leq \frac{\max_{t\in T} |f^{-1}(t)|}{\min_{s\in S} |f(s)|}.$$

Lemma 3.2. *For any natural numbers $l \leq k$ the following inequality holds:*

$$\frac{S(k, l-1)}{S(k, l)} \leq \frac{l(l-1)}{2(k-l+1)}.$$

Proof. Let $\mathbb{A}_k^l$ denote the set of all partitions of $[k]$ into l blocks. Consider a multi-valued map $\tau : \mathbb{A}_k^l \to \mathbb{A}_k^{l-1}$ which takes a partition $p \in \mathbb{A}_k^l$ and glues any two of its blocks. It is clear that every element in $\mathbb{A}_k^l$ has $\binom{l}{2}$ images. Now, suppose a partition $p \in \mathbb{A}_k^{l-1}$ has blocks of sizes $x_1, x_2, \ldots, x_{l-1}$. Then $|f^{-1}(p)|$ is equal to $\sum_{s=1}^{l-1}(2^{x_s-1} - 1)$. Indeed, $|f^{-1}(p)|$ is simply the number of ways to split one of the blocks of p into two, and the number of ways to split a block of size x into two is given by $2^{x-1} - 1$. Now note that $\sum_{s=1}^{l-1}(2^{x_s-1} - 1) \geq \sum_{s=1}^{l-1}(x_s - 1) = k - l + 1$, thus the multi-valued map principle gives us

$$\frac{S(k, l-1)}{S(k, l)} = \frac{|\mathbb{A}_k^{l-1}|}{|\mathbb{A}_k^l|} \leq \frac{l(l-1)}{2(k-l+1)}.$$

□

Remark 1. Note that the inequality is asymptotically tight for $l = o(\sqrt{k})$, see [4].

We shall sometimes need a weaker bound given by the following trivial corollary.

Corollary 3.2. *For any natural numbers $l \leq k$ the following inequality is valid:*

$$\frac{S(k, l-1)}{S(k, l)} \leq \frac{k^2}{2}.$$

In the proof of Theorem 7 we use the following lemma, see [8, Corollary 5].

Lemma 3.3. *Suppose we have sequences k_i, n_i. In the following we omit indexes to lighten the notation. Assume that $k/n = c + o(1)$, with $c \in (0, 1)$. Then the following asymptotics for $S(n, k)$ holds:*

$$S(n, k) = n^{n-k} \cdot e^{g(c)\cdot n + o(n)},$$

where $g(c) = c + \log \gamma + (\gamma - c) \cdot \log (\gamma - c) - \gamma \cdot \log \gamma$ and γ is the unique solution of $\gamma \cdot (1 - e^{-1/\gamma}) = c$.

Remark 2. In [8] much tighter asymptotic expansion is given for $k = cn + o(n^{2/3})$. In order to pass to the case $k = cn + o(n)$ we can use Lemma 3.2.

4 Supremum of several partitions

We now turn to studying the supremum of several randomly chosen partitions. Suppose maps $\phi_1, \phi_2, \ldots, \phi_t : [n] \to [n]$ are chosen independently according to the uniform measure on the set of all maps. We are interested in the question of how likely $p := \sup_{1\le i\le t} p_{\phi_i}$ is to be equal to $p_{\max} = \{[n]\}$. Here the threshold value of t equals $\log(n)$ where log denotes the natural logarithm. That is, if $t = t(n) = \log(n) - w(n)$ with $w(n) \to \infty$ arbitrarily slowly then $\mathbb{P}[p = p_{\max}] \to 0$; whereas if $t = t(n) = \log(n) + w(n)$ then $\mathbb{P}[p = p_{\max}] \to 1$. We start with the following technical result which shall be used several times.

Lemma 4.1. *Let M be the number of one-element blocks in $\sup\{p_{\phi_1}, \ldots, p_{\phi_t}\}$. Then*

$$\mathbb{E}[M] = ne^{-t} + O(te^{-t}), \qquad \mathrm{Var}(M) = O\left(\max\{nte^{-2t}, te^{-t}\}\right), \qquad n \to \infty,$$

where both $O(\cdot)$ are uniform in $t = 1, \ldots, 2\log n$.

Proof. By the linearity of the expectation and due to the symmetry,

$$\mathbb{E}[M] = n \cdot \mathbb{P}[\{1\} \text{ forms a one-element block in } p] = n \cdot \left(\left(1 - \frac{1}{n}\right)^{n-1}\right)^t.$$

So

$$\mathbb{E}[M] - ne^{-t} = n\left[\left(1 - \frac{1}{n}\right)^{n-1} - e^{-1}\right]\sum_{s=0}^{t-1}\left(1 - \frac{1}{n}\right)^{s(n-1)} e^{s-t+1}.$$

As $n \to \infty$, the expression in brackets is of order $O(1/n)$ while each summand is bounded above by $e^{-t+1}\left(1 - \frac{1}{n}\right)^{-s} \sim e^{-t+1}$ for $s < t \le 2\log n$. Similarly,

$$\begin{aligned}\mathbb{E}\left[\binom{M}{2}\right] &= \binom{n}{2} \cdot \mathbb{P}[\{1\} \text{ and } \{2\} \text{ are two one-element blocks in } p] \\ &= \binom{n}{2} \cdot \left(\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)^{n-2}\right)^t = \binom{n}{2} \cdot e^{-2t} + O(nte^{-2t}), \qquad n \to \infty.\end{aligned}$$

Hence

$$\mathrm{Var}(M) = 2 \cdot \mathbb{E}\left[\binom{M}{2}\right] + \mathbb{E}[M] - (\mathbb{E}[M])^2 = O\left(\max\{nte^{-2t}, te^{-t}\}\right).$$

□

Now we are ready to formulate and prove the result for the case $t - \log(n) \to -\infty$.

Theorem 3. *Let $w : \mathbb{N} \to \mathbb{R}$ be a function such that $\lim_{n\to\infty} w(n) = \infty$ and $w(n) < \log(n)$. Let $t = t(n) = \log(n) - w(n)$ be an integer, and suppose maps $\phi_1, \phi_2, \ldots, \phi_t : [n] \to [n]$ are chosen independently according to the uniform measure on the set of all maps. Then*

$$\lim_{n\to\infty} \mathbb{P}[\sup\{p_{\phi_1}, \ldots, p_{\phi_t}\} = p_{\max}] = 0.$$

Proof. We denote $\sup\{p_{\phi_1}, \ldots, p_{\phi_t}\}$ by p. Let M be the number of one-element blocks in p. We want to show that $\mathbb{P}[M = 0]$ tends to zero as n tends to infinity. In order to do this we plug $t = \log n - w(n)$ into the expressions of Lemma 4.1 to find out that

$$\mathbb{E}[M] = e^{w(n)}\left(1 + O\left(\frac{\log n}{n}\right)\right), \qquad \mathrm{Var}(M) = O\left(\frac{\log n}{n} e^{2w(n)}\right), \qquad n \to \infty.$$

So we can use the Chebyshev inequality to bound the probability that M equals zero:

$$\mathbb{P}[M = 0] \le \frac{\mathrm{Var}(M)}{(\mathbb{E}[M])^2} = O\left(\frac{\log n}{n}\right) + \left(1 + O\left(\frac{\log n}{n}\right)\right) \cdot e^{-w(n)} \to 0, \qquad n \to \infty.$$

□

In order to prove that for $t = \log(n) + w(n)$ the partition $p := \sup\{p_{\phi_1}, \ldots, p_{\phi_t}\}$ is likely to be equal to $p_{\max}$ we need the following three lemmas. The first lemma claims that blocks of size less than $c \cdot \sqrt{n}$ are unlikely to appear in p; the second lemma claims that blocks of size between $c \cdot \sqrt{n}$ and $\log n \cdot \sqrt{n}$ are also unlikely to appear in p. Finally, the third lemma claims that p is unlikely to have two blocks of size at least $\log n \cdot \sqrt{n}$.

Lemma 4.2. *There exist an absolute constant $c > 0$ such that the following holds. Let $w : \mathbb{N} \to \mathbb{R}$ be a function such that $\lim_{n\to\infty} w(n) = \infty$ and $w(n) < \log(n)$. Let $t = t(n) = \log(n) + w(n)$ be an integer, and suppose maps $\phi_1, \phi_2, \ldots, \phi_t : [n] \to [n]$ are chosen independently according to the uniform measure on the set of all maps. Let $p := \sup\{p_{\phi_1}, \ldots, p_{\phi_t}\}$, then*

$$\mathbb{P}\left[p \textit{ has a block of size at most } c \cdot \sqrt{n}\right] < 9 \cdot e^{-w(n)}.$$

Proof. We may assume that n is large enough for our argument to work. We fix $k \le c \cdot \sqrt{n}$ with small enough c, say $c = \frac{1}{100}$, and bound the probability that p has a block of size k:

$$\begin{aligned}\mathbb{P}[p \text{ has a block of size } k] &\le \binom{n}{k} \cdot \mathbb{P}[\{1, 2, \ldots, k\} \text{ is a block of } p] \\ &\le \binom{n}{k} \left(\mathbb{P}[\phi(a) \neq \phi(b) \text{ for any } a \le k < b]\right)^t.\end{aligned}$$

Note that for a fixed l, the number of maps ϕ such that $\phi(a) \neq \phi(b)$ for any $a \le k < b$ and the image of $\{1, 2, \ldots, k\}$ under ϕ has l elements, equals $\binom{n}{l} \cdot l! \cdot S(k, l) \cdot (n - l)^{n-k}$. Thus this expression can be rewritten in terms of Stirling numbers of the second kind as follows:

$$\begin{aligned}\mathbb{P}[p \text{ has a block of size } k] &\le \binom{n}{k} \left(\sum_{l=1}^{k} \frac{1}{n^n} \cdot \binom{n}{l} \cdot l! \cdot S(k, l) \cdot (n - l)^{n-k}\right)^t \\ &\le \binom{n}{k} \left(\sum_{l=1}^{k} n^{l-n} \cdot S(k, l) \cdot (n - l)^{n-k}\right)^t.\end{aligned} \tag{1}$$

We now estimate the sum in parentheses. Denoting $n^{l-n} \cdot S(k,l) \cdot (n-l)^{n-k}$ by $f_k(l)$, we have, for $k \le c \cdot \sqrt{n}$

$$f_k(k) = \left(1 - \frac{k}{n}\right)^{n-k} \le e^{-k}\left(1 - \frac{k}{n}\right)^{-k} \le e^{-k}\left(1 + \frac{2k^2}{n}\right).$$

Now we want to show that as l decreases from k to 1, $f_k(l)$ decreases fast enough. Namely, we have, for $2 \le l \le k$

$$\frac{f_k(l-1)}{f_k(l)} = \frac{1}{n} \cdot \frac{S(k,l-1)}{S(k,l)} \cdot \left(1 + \frac{1}{n-l}\right)^{n-k} \le \frac{e \cdot k^2}{2n}. \tag{2}$$

Here the last inequality is due to Corollary 3.2. Putting this together we obtain

$$\begin{aligned}
\mathbb{P}[p \text{ has a block of size } k] &\le \binom{n}{k}\left(\sum_{l=1}^{k} n^{l-n} \cdot S(k,l) \cdot (n-l)^{n-k}\right)^t \\
&\le \binom{n}{k}\left(\sum_{s=0}^{k-1} e^{-k}\left(1 + \frac{2k^2}{n}\right) \cdot \left(\frac{e \cdot k^2}{2n}\right)^s\right)^t \\
&\le \frac{n^k}{k!}\left(e^{-k}\left(1 + \frac{5k^2}{n}\right)\right)^t \\
&\le e^{-k \cdot w(n)} \cdot \frac{1}{k!} \cdot \left(1 + \frac{5k^2}{n}\right)^{2\log n} \le \frac{5}{k^2} \cdot e^{-w(n)}.
\end{aligned}$$

The last inequality is valid for all sufficiently large n and any $k \le c \cdot \sqrt{n}$. Indeed, if $k < n^{1/4}$, we argue that $\left(1 + \frac{5k^2}{n}\right)^{2\log n} \le 2$ for sufficiently large n and the inequality follows immediately. Otherwise, if $n^{1/4} \le k \le c \cdot \sqrt{n}$, for sufficiently large n we have $\frac{1}{k!} \cdot \left(1 + \frac{5k^2}{n}\right)^{2\log n} \le \frac{1}{k!} \cdot 2^{2\log n} < 1$.

Summing over all possible $k \le c \cdot \sqrt{n}$ we deduce that

$$\mathbb{P}\left[p \text{ has a block of size at most } c \cdot \sqrt{n}\right] < 9 \cdot e^{-w(n)}.$$

□

Lemma 4.3. *For any constant $c > 0$ there exists an absolute constant $C > 0$ such that the following holds. Let $w : \mathbb{N} \to \mathbb{R}$ be a function such that $\lim_{n\to\infty} w(n) = \infty$ and $w(n) < \log(n)$. Let $t = t(n) = \log(n) + w(n)$ be an integer, and suppose maps $\phi_1, \phi_2, \ldots, \phi_t : [n] \to [n]$ are chosen independently according to the uniform measure on the set of all maps. Let $p := \sup\{p_{\phi_1}, \ldots, p_{\phi_t}\}$, then*

$$\mathbb{P}\left[p \text{ has a block of size at least } c \cdot \sqrt{n} \text{ and at most } \log n \cdot \sqrt{n}\right] < C \cdot e^{-n^{1/2}}.$$

Proof. We may assume that n is large enough. Let us fix k between $c\cdot\sqrt{n}$ and $\log(n)\cdot\sqrt{n}$ and estimate the probability that p has a block of size k. Similarly to (1) we have

$$\mathbb{P}[p \text{ has a block of size } k] \leq \binom{n}{k}\left(\sum_{l=1}^{k} n^{l-n}\cdot S(k,l)\cdot(n-l)^{n-k}\right)^t.$$

We now estimate the sum in parentheses, though this time slightly differently. We again denote $n^{l-n}\cdot S(k,l)\cdot(n-l)^{n-k}$ by $f_k(l)$. We have

$$f_k(k) = \left(1-\frac{k}{n}\right)^{n-k} \leq e^{-k}\cdot\left(1-\frac{k}{n}\right)^{-k} \leq e^{-k}\cdot e^{2k^2/n}.$$

Now we want to show that as l decreases from k to 1, $f_k(l)$ does not increase for too long. Namely, we have, for $2\leq l\leq k$

$$\frac{f_k(l-1)}{f_k(l)} = \frac{1}{n}\cdot\frac{S(k,l-1)}{S(k,l)}\cdot\left(1+\frac{1}{n-l}\right)^{n-k} \leq \frac{e}{n}\cdot\frac{S(k,l-1)}{S(k,l)} \leq \frac{e\cdot k^2}{2n\cdot(k-l+1)},$$

where the last inequality is due to Lemma 3.2. It is clear from this inequality that $f_k(l-1)\leq f_k(l)$ for all $l\leq k-\frac{2k^2}{n}$. Thus, $\max_l\{f_k(l)\}$ is bounded by $f_k(k)\cdot\left(\frac{e\cdot s}{2}\right)^{2s}$, where $s := \frac{k^2}{n} \leq (\log n)^2$. We now use this bound to obtain

$$\begin{aligned}
\mathbb{P}[p \text{ has a block of size } k] &\leq \binom{n}{k}\left(\sum_{l=1}^{k} n^{l-n}\cdot S(k,l)\cdot(n-l)^{n-k}\right)^t \\
&= \binom{n}{k}\left(\sum_{l=1}^{k} f_k(l)\right)^t \leq \binom{n}{k}\left(k\cdot\max_l\{f_k(l)\}\right)^t \\
&\leq \frac{n^k}{k!}\left(e^{-k+2k^2/n}\cdot k\cdot(2s)^{2s}\right)^t \\
&= \frac{1}{k!}\left(e^{2s}\cdot k\cdot(2s)^{2s}\right)^t\cdot\frac{n^k}{e^{kt}} \\
&\leq \frac{1}{e^{k\log k/2}}\left(C_1\cdot n\cdot e^{s^2}\right)^t \\
&\leq \frac{1}{e^{c\sqrt{n}\log n/2}}\left(C_1\cdot n\cdot e^{(\log n)^4}\right)^{2\log n} \\
&\leq e^{-2\sqrt{n}+4\cdot(\log n)^5} \\
&\leq \frac{1}{n}\cdot e^{-n^{1/2}}.
\end{aligned}$$

The above estimate is valid for sufficiently large n and some absolute constant C_1. Note that we used the fact that $\frac{(2se)^{2s}}{e^{s^2}}$ is bounded on $\mathbb{R}_+$. Finally, summing over all possible $k\leq\log n\cdot\sqrt{n}$ we deduce that for some constant C

$$\mathbb{P}\left[p \text{ has a block of size at least } c\cdot\sqrt{n} \text{ and at most } \log n\cdot\sqrt{n}\right] < C\cdot e^{-n^{1/2}}.$$

□

Lemma 4.4. *Let p' be a fixed partition of $[n]$ with all blocks of size at least $\log n \cdot \sqrt{n}$. If the map $\phi : [n] \to [n]$ is chosen randomly according to the uniform measure on the set of all maps, then for n large enough*

$$\mathbb{P}[\sup\{p_\phi, p'\} \neq p_{\max}] < n^2 \cdot e^{-(\log n)^2/2}.$$

Proof. Note that if $\sup\{p_\phi, p'\} \neq p_{\max}$ then in p' there exist two blocks $\{x_1, x_2, \ldots, x_a\}$ and $\{y_1, y_2, \ldots, y_b\}$ such that p_ϕ does not 'merge' these two blocks, that is, $\phi(x_i) \neq \phi(y_j)$ for any $i \le a$ and $j \le b$. We now want to show that the probability of such event is small for any two fixed blocks. It is sufficient to consider the case when both blocks have size $t = \log n \cdot \sqrt{n}$. Since the number of blocks in p' is at most $\sqrt{n}$, the union bound gives us

$$\mathbb{P}[\sup\{p_\phi, p'\} \neq p_{\max}] \le n \cdot \mathbb{P}[\phi(\{1,2,\ldots,t\}) \cap \phi(\{t+1, t+2, \ldots, 2t\}) = \emptyset]. \qquad (3)$$

The probability on the right-hand side equals

$$\frac{1}{n^t} \sum_{k=1}^{t} \binom{n}{k} \cdot S(t,k) \cdot k! \cdot \left(1 - \frac{k}{n}\right)^t \le \sum_{k=1}^{t} n^{k-t} \cdot S(t,k) \cdot \left(1 - \frac{k}{n}\right)^t.$$

Let us denote $n^{k-t} \cdot S(t,k) \cdot \left(1 - \frac{k}{n}\right)^t$ by $s_t(k)$. Then $s_t(t) = \left(1 - \frac{t}{n}\right)^t$ and

$$\frac{s_t(k-1)}{s_t(k)} = \frac{1}{n} \cdot \frac{S(t,k-1)}{S(t,k)} \cdot \left(1 + \frac{1}{n-k}\right)^t \le \frac{2}{n} \cdot \frac{t^2}{2(t-k+1)},$$

where the last inequality is due to Lemma 3.2. This quantity is less than 1 for all $k < t - (\log n)^2$, hence the maximal value $\max_l\{s_t(l)\}$ is achieved for some $k > t - (\log n)^2$. We thus have the following estimate:

$$\begin{aligned}
\max_{1 \le l \le t}\{s_t(l)\} = s_t(k) &= s_t(t) \cdot \frac{s_t(t-1)}{s_t(t)} \cdots \frac{s_t(k)}{s_t(k+1)} \le s_t(t) \cdot \prod_{l=k+1}^{t} \left(\frac{t^2}{n} \cdot \frac{\left(1 + \frac{1}{n-l}\right)^t}{2(t-l+1)} \right) \\
&= \left(1 - \frac{t}{n}\right)^t \cdot \left(1 + \frac{t-k}{n-t}\right)^t \cdot (\log n)^{2(t-k)} \cdot \frac{1}{2^{t-k} \cdot (t-k)!} \\
&\le \left(1 - \frac{k}{n}\right)^t \cdot \left(\frac{e \cdot (\log n)^2}{2(t-k)}\right)^{t-k} = (1 + o(1)) \cdot e^{-(\log n)^2} \cdot \left(\frac{e \cdot (\log n)^2}{2(t-k)}\right)^{t-k},
\end{aligned}$$

as $n \to \infty$. Define x to be $\frac{t-k}{(\log n)^2}$. Then $x \le 1$ and we have

$$s_t(k) \le 2 \cdot e^{-(\log n)^2} \cdot \left(\frac{e \cdot (\log n)^2}{2(t-k)}\right)^{t-k} = 2 \cdot \left(\frac{\left(\frac{e}{2x}\right)^x}{e}\right)^{(\log n)^2} \le 2 \cdot e^{-(\log n)^2/2}$$

because $\left(\frac{e}{2x}\right)^x \le \sqrt{e}$ for $x \ge 0$. Now it follows immediately that the probability on the right-hand side of (3) is at most $2t \cdot e^{-(\log n)^2/2}$, hence

$$\mathbb{P}[\sup\{p_\phi, p'\} \neq p_{\max}] \le n \cdot 2t \cdot e^{-(\log n)^2/2} < n^2 \cdot e^{-(\log n)^2/2}.$$

□

We now prove that if we have substantially more than $\log n$ partitions then their supremum is likely to be equal to $p_{\max}$.

Theorem 4. *Let $w : \mathbb{N} \to \mathbb{R}$ be a function such that $\lim_{n\to\infty} w(n) = \infty$. Let $t = t(n) = \log(n) + w(n)$ be an integer, and suppose maps $\phi_1, \phi_2, \ldots, \phi_t : [n] \to [n]$ are chosen independently according to the uniform measure on the set of all maps. Then*

$$\lim_{n\to\infty} \mathbb{P}[\sup\{p_{\phi_1}, \ldots, p_{\phi_t}\} = p_{\max}] = 1.$$

Proof. We may assume that $w(n) < \log n$ since the more partitions we take the more likely their supremum is to be $p_{\max}$. Lemmas 4.2 and 4.3 show that the partition $p' := \sup\{p_{\phi_2}, \ldots, p_{\phi_t}\}$ has no blocks of size less than $\log n \cdot \sqrt{n}$ with probability $1 - o(1)$. Hence, by Lemma 4.4 we have $p = \sup\{p', p_{\phi_1}\} = p_{\max}$ with probability $1 - o(1)$. □

5 The size of the largest block

In this section we study the typical picture for $p = \sup\{p_{\phi_1}, p_{\phi_2}, \ldots, p_{\phi_t}\}$ when t is fixed. For $t = 3, 4$, p is likely to have a block of size $\Omega(n)$, as shown in Theorems 7 and 5, the former requiring much more subtle asymptotics for $S(n, k)$. Theorem 6 claims that for larger t, the partition p is likely to have a block of size $n - \varepsilon_t \cdot n$, where ε_t decays exponentially in t. We also show in Theorem 8 that contrary to the case $t = 3$, if we consider a supremum of two random partitions, it is likely to have no blocks of size $\Omega(n)$.

For further results we need the notion of *k–free* partition. For any $k < n$ define the set of partitions E_k to be $\{p|$ there is no partition $p' \succeq p$ having a block of size $k\}$. We shall call partitions from the set E_k *k–free* partitions. We first formulate several simple properties of k–free partitions and prove them.

Lemma 5.1. *Suppose a partition p of $[n]$ is k–free for any $k \in [a, b]$ where $a < b$ are natural numbers, then p has a block of size at least $b - a$.*

Proof. Arguing by contradiction we suppose that p has blocks of sizes $x_1 \le x_2 \le \ldots x_r < b - a$. Let s be the first index such that $x_1 + \cdots + x_s \ge a$, then $a \le x_1 + \cdots + x_s = (x_1 + \cdots + x_{s-1}) + x_s \le a + (b - a) = b$, which is a contradiction as p is not $x_1 + \cdots + x_s$–free. □

Lemma 5.2. *Suppose a partition p of $[n]$ is k–free for any $k \in [a, b]$ where a, b are natural numbers satisfying $2 \cdot a \le b$ then p has a block of size at least b.*

Proof. By Lemma 5.1 p has a block of size at least $b - a \ge a$. Clearly p cannot have a block of size $k \in [a, b]$ and thus p has a block of size at least b. □

Lemma 5.3. *Suppose a partition p of $[n]$ has h blocks of size 1 and b–free for some $b > h$, then p has a block of size at least h.*

Proof. Consider a partition p' of a set with $n-h$ elements obtained by removing all blocks of size 1 from p. It is easy to see that p' is k–free for any $k \in [b-h, b]$. Indeed, if a union of some blocks in p' had size $b-k$ for some $k \leq h$ then adding k blocks of size 1 we would deduce that a union of some blocks in p has size b which is impossible. By Lemma 5.1 p' has a block of size at least h. Hence, p has a block of size at least h. $\square$

Lemma 5.4. *Suppose a partition p of $[n]$ is k–free for any $k \in [a, b]$ where a, b are natural numbers satisfying $2 \cdot a \leq b$. Then the size of the union of all blocks in p which have size at least a is bounded below by $n - a$.*

Proof. We argue by contradiction. We shall call blocks of size at least a *big* and all other blocks *small*. Suppose the union of all big blocks has size smaller than $n - a$, then the union of small blocks has size greater than a. Let $x_1, x_2, \ldots, x_r$ be the sizes of small blocks, thus $x_1 + x_2 + \cdots + x_r > a$. Consider the smallest index i such that $x_1 + \cdots + x_i \geq a$, then we have $a \leq x_1 + x_2 + \cdots + x_i = (x_1 + \ldots x_{i-1}) + x_i \leq a + a \leq b$ which is a contradiction as p cannot be $(x_1 + \cdots + x_i)$–free. $\square$

Theorem 5. *Suppose four maps $\phi_1, \phi_2, \phi_3, \phi_4 : [n] \to [n]$ are chosen independently according to the uniform measure. Let $p = \sup\{p_{\phi_1}, p_{\phi_2}, p_{\phi_3}, p_{\phi_4}\}$, then*

$$\lim_{n\to\infty} \mathbb{P}\left[p \textit{ has a block of size at least } \frac{n}{3}\right] = 1.$$

Proof. Let $c \in [\frac{1}{11}, \frac{1}{3}]$ be a constant and let $k = c \cdot n$. We shall see that p is exponentially unlikely to be in $\Pi_n \setminus E_k$, and thus with high probability it lies in E_k. Similarly to the proof of Lemma 4.2 we have

$$\begin{aligned}
\mathbb{P}[p \notin E_k] &\leq \binom{n}{k} \left(\mathbb{P}[\phi(a) \neq \phi(b) \text{ for any } a \leq k < b]\right)^4 \\
&= \binom{n}{k} \left(\sum_{l=1}^{k} \frac{1}{n^n} \cdot \binom{n}{l} \cdot l! \cdot S(k,l) \cdot (n-l)^{n-k}\right)^4 \\
&\leq \binom{n}{k} \left(\sum_{l=1}^{k} n^{l-k} \cdot S(k,l) \cdot \left(1 - \frac{l}{n}\right)^{n-k}\right)^4 .
\end{aligned}$$

We again denote $n^{l-k} \cdot S(k,l) \cdot \left(1 - \frac{l}{n}\right)^{n-k}$ by $f_k(l)$. We want to bound $\max_l\{f_k(l)\}$. Note that due to Lemma 3.2 we have

$$\frac{f_k(l-1)}{f_k(l)} = \frac{1}{n} \cdot \frac{S(k,l-1)}{S(k,l)} \cdot \left(1 + \frac{1}{n-l}\right)^{n-k} \leq \frac{e \cdot l^2}{2 \cdot n \cdot (k-l+1)},$$

thus the maximum of $f_k(l)$ over l for fixed k, n must be achieved for some $l \geq k/2$. We now bound $f_k(l)$.

$$f_k(l) = f_k(k) \cdot \prod_{r=l+1}^{k} \left(\frac{f_k(r-1)}{f_k(r)}\right) \leq f_k(k) \cdot \prod_{r=l+1}^{k} \left(\frac{e \cdot r^2}{2 \cdot n \cdot (k-r+1)}\right).$$

Note that the factors which correspond to $r < k - x \cdot n$ are smaller than 1, where x is the smallest solution of the equation

$$2 \cdot x = e \cdot (c - x)^2. \tag{4}$$

We can then disregard all these factors to deduce that

$$\begin{aligned}\sum_{l=1}^{k} f_k(l) &\leq k \cdot \max_l \{f_k(l)\} \\ &\leq k \cdot f_k(k) \cdot \prod_{r=(c-x)\cdot n}^{c\cdot n} \left(\frac{e \cdot r^2}{2 \cdot n \cdot (k - r + 1)} \right) \\ &= k \cdot \left(1 - \frac{k}{n}\right)^{n-k} \cdot \left(\frac{e}{2 \cdot n}\right)^{xn} \cdot \left(\frac{(cn)!}{((c-x)\cdot n)!} \right)^2 \cdot \frac{1}{(xn)!}.\end{aligned}$$

We now take log and divide through by n to obtain

$$\begin{aligned}\frac{\log\left(\max_l\{f_k(l)\}\right)}{n} \leq{}& (1-c) \cdot \log(1-c) + x \cdot (1 - \log 2) - x \cdot \log n + 2 \cdot c \log(cn) - 2 \cdot c \\ &- 2 \cdot (c-x) \log((c-x)n) + 2 \cdot (c-x) - x \cdot \log(xn) + x + o(1).\end{aligned}$$

We see that all summands of order $\log n$ magically cancel out and we obtain

$$\begin{aligned}\frac{\log\left(\sum_{l=1}^{k} f_k(l)\right)}{n} &\leq (1-c) \cdot \log(1-c) + x \cdot (2 - \log 2) + 2 \cdot c \log c - 2 \cdot c \\ &\quad - 2 \cdot (c-x) \log(c-x) + 2 \cdot (c-x) - x \cdot \log x + o(1) \\ &= (1-c) \cdot \log(1-c) - x \cdot \log 2 + 2 \cdot c \log c \\ &\quad - 2 \cdot (c-x) \log(c-x) - x \cdot \log x + o(1).\end{aligned}$$

We denote the right-hand side by $\mu(c)$. Note that it indeed depends on c only, as x can be expressed in terms of c using (4). We have now the following estimate for $\mathbb{P}[p \notin E_k]$:

$$\begin{aligned}\mathbb{P}[p \notin E_k] &\leq \binom{n}{k} \cdot \left(\sum_{l=1}^{k} n^{l-k} \cdot S(k,l) \cdot \left(1 - \frac{l}{n}\right)^{n-k} \right)^4 \\ &\leq e^{n\cdot(H(c) + 4\mu(c) + o(1))},\end{aligned}$$

where we use the standard notation for the entropy function:

$$H(c) = -c \cdot \log c - (1-c) \cdot \log(1-c). \tag{5}$$

Thus $\mathbb{P}[p \notin E_k]$ decays exponentially whenever $\lambda(c) := H(c) + 4 \cdot \mu(c) < 0$ which turns out to be the case for $c \in [0.087412, 0.340034]$. Moreover, since $\lambda(c)$ is continuous, there exists $\varepsilon > 0$ such that $\lambda(c) < -\varepsilon$ for all $c \in [1/11, 1/3]$. Hence, the union bound gives us

$$\mathbb{P}\left[p \notin E_k \text{ for some } k \in \left[\frac{n}{11}, \frac{n}{3}\right]\right] \leq n \cdot e^{-\varepsilon \cdot n}.$$

It follows from Lemma 5.2 that p has a block of size at least $\frac{n}{3}$ with probability at least $1 - n \cdot e^{-\varepsilon \cdot n}$. □

Theorem 6. *For any $\varepsilon \in (0, \log(e/2))$ there exists a constant $C > 0$ such that for any fixed $t > C$ the following holds. Suppose maps $\phi_1, \phi_2, \ldots, \phi_t : [n] \to [n]$ are chosen independently according to the uniform measure. Let $p = \sup\{p_{\phi_1}, p_{\phi_2}, \ldots, p_{\phi_t}\}$ and denote the size of the largest block in p by L, then*

$$\lim_{n\to\infty} \mathbb{P}\left[1 - \tfrac{1}{2}e^{-t+3-\varepsilon} \le \frac{L}{n} \le 1 - e^{-t-\varepsilon}\right] = 1.$$

Proof. We first prove that $\lim_{n\to\infty} \mathbb{P}\left[L < n - n \cdot e^{3-t}\right] = 0$. The proof is similar to the one of Theorem 5, though this time the calculations are easier. We consider the partition $p' = \sup\{p_{\phi_2}, p_{\phi_3}, \ldots p_{\phi_t}\}$. Take $k = n \cdot e^{c-t}$ for some $c \in [2+\delta, 3]$, with $\delta > 0$ small and fixed, we again have

$$\mathbb{P}[p' \notin E_k] \le \binom{n}{k} \left(\sum_{l=1}^{k} n^{l-k} \cdot S(k,l) \cdot \left(1 - \frac{l}{n}\right)^{n-k} \right)^{t-1}.$$

This time we write the following rougher bound, for $1 \le l \le k$

$$\frac{f_k(l-1)}{f_k(l)} = \frac{1}{n} \cdot \frac{S(k,l-1)}{S(k,l)} \cdot \left(1 + \frac{1}{n-l}\right)^{n-k} \le \frac{2 \cdot k^2}{n \cdot (k-l+1)},$$

and this quantity is smaller than 1 for $l < k - \frac{2 \cdot k^2}{n}$. Hence we obtain the following bound:

$$\begin{aligned}
\sum_{l=1}^{k} f_k(l) &\le k \cdot \max_l \{f_k(l)\} \\
&\le k \cdot f_k(k) \cdot \prod_{r=k-k^2/n}^{k} \left(\frac{2k^2}{n \cdot (k-r+1)} \right) \\
&\le k \cdot \left(1 - \frac{k}{n}\right)^{n-k} \cdot 2^{k^2/n} \cdot e^{k^2/n} \\
&\le e^{-k+5k^2/n}.
\end{aligned}$$

Let us denote $\frac{k}{n} = e^{c-t}$ by α. Recalling the notation $H(\alpha)$ from (5) we have the following inequality:

$$\mathbb{P}[p' \notin E_k] \le \binom{n}{k} \left(\sum_{l=1}^{k} f_k(l) \right)^{t-1} \le e^{n \cdot H(\alpha)} \cdot e^{-(t-1) \cdot n \cdot (\alpha - 5 \cdot \alpha^2)}.$$

In order to conclude the proof we use an estimate $H(\alpha) \le -\alpha \log \alpha + \alpha$ which allows us to write

$$\begin{aligned}
H(\alpha) - (t-1) \cdot (\alpha - 5 \cdot \alpha^2) &\le (-\alpha \cdot \log \alpha + \alpha - (t-1) \cdot (\alpha - 5 \cdot \alpha^2)) \\
&= ((2-c) + 5 \cdot (t-1) \cdot \alpha) \cdot \alpha \le \frac{-\delta \cdot \alpha}{2}.
\end{aligned}$$

The last inequality is valid for sufficiently large t, since $2-c<-\delta$ and $\alpha=e^{c-t}$, so $t\cdot\alpha$ vanishes as $t\to\infty$. Consequently, the union bound gives us an estimate for the probability that $p'\in E_k$ for some $k\in[e^{2+\delta-t}\cdot n, e^{3-t}\cdot n]$:

$$\mathbb{P}[\text{there exists } k\in[e^{2+\delta-t}\cdot n, e^{3-t}\cdot n] \text{ such that } p'\notin E_k]\le e^{-\delta\alpha\cdot n/4}.$$

We denote $\cap_{k\in[e^{2+\delta-t}\cdot n, e^{3-t}\cdot n]}E_k$ by E. The above statement claims that $\mathbb{P}[p'\notin E]\le e^{-\delta\alpha\cdot n/4}$. For $\delta<1-\log 2$, say for $\delta=1-\log 2-\varepsilon$ with ε mentioned in the claim, by Lemma 5.4 we know that for any $p'\in E$ the union of all blocks of size at least $c_1:=e^{2+\delta-t}=\frac{1}{2}e^{3-t-\varepsilon}$ in p' has size at least $(1-c_1)\cdot n$. Finally the argument presented in the proof of Lemma 4.4 shows that in $p=\sup\{p',p_1\}$ all these blocks are merged with probability $1-o(1)$ and thus p has a block of size at least $(1-c_1)\cdot n$ with probability tending to 1.

We now prove that $\lim_{n\to\infty}\mathbb{P}\left[L>n-n\cdot e^{-t-\varepsilon}\right]=0$. The argument is very similar to the one presented in the proof of Theorem 3. Let M again be the number of one-element blocks in p. As $n\to\infty$ with t fixed by Lemma 4.1 we have $\mathbb{E}[M]=e^{-t}\cdot n\cdot(1+O(1/n))$ and $\mathrm{Var}(M)=O(n)$. Trivially, the largest block has size $n-M$ at most, thus by Chebyshev's inequality

$$\mathbb{P}\left[L>n-n\cdot e^{-t-\varepsilon}\right]\le\mathbb{P}\left[M<n\cdot e^{-t-\varepsilon}\right]\le\frac{\mathrm{Var}(M)}{\left(\mathbb{E}[M]-n\cdot e^{-t-\varepsilon}\right)^2}=O\left(\frac{1}{n}\right).$$

□

In order to prove Theorem 7 we need the following lemma which claims that the supremum of three random partitions is likely to have $\Omega(n)$ one-element blocks.

Lemma 5.5. *Suppose three maps $\phi_1,\phi_2,\phi_3:[n]\to[n]$ are chosen independently according to the uniform measure. Let $p=\sup\{p_{\phi_1},p_{\phi_2},p_{\phi_3}\}$, then for any constant $c\in(0,e^{-3})$*

$$\lim_{n\to\infty}\mathbb{P}\left[p \textit{ has at least } c\cdot n \textit{ one-element blocks}\right]=1.$$

Proof. The proof uses the same technique as presented in the proof of Theorem 3. Let M denote the number of one-element blocks in p. By Lemma 4.1 we have, as $n\to\infty$, $\mathbb{E}[M]=e^{-3}\cdot n+O(1)$, $\mathrm{Var}(M)=O(n)$. So, by Chebyshev's inequality, the probability that M is less than $c\cdot n$ is

$$\mathbb{P}[M<c\cdot n]\le\frac{\mathrm{Var}(M)}{(\mathbb{E}[M]-c\cdot n)^2}=O\left(\frac{1}{n}\right).$$

□

Theorem 7. *There exists a constant $c>0$ such that the following holds. Suppose three maps $\phi_1,\phi_2,\phi_3:[n]\to[n]$ are chosen independently according to the uniform measure. Let $p=\sup\{p_{\phi_1},p_{\phi_2},p_{\phi_3}\}$, then*

$$\lim_{n\to\infty}\mathbb{P}\left[p \textit{ has a block of size at least } c\cdot n\right]=1.$$

Proof. In the light of Lemma 5.5 and Lemma 5.3 it is sufficient to show that $\mathbb{P}[p \in E_{n/2}]$ tends to 1 as n grows to infinity. Similarly to the proof of Theorem 5 we have

$$\begin{aligned}\mathbb{P}[p \notin E_{n/2}] &\leq \binom{n}{n/2} (\mathbb{P}[\phi(a) \neq \phi(b) \text{ for any } a \leq n/2 < b])^3 \\ &= \binom{n}{n/2} \left(\sum_{l=1}^{n/2} \frac{1}{n^n} \cdot \binom{n}{l} \cdot l! \cdot S(n/2, l) \cdot (n-l)^{n/2} \right)^3 \\ &\leq n^2 \cdot \left(\sum_{l=1}^{n/2} \left(2^{n/3} \cdot \frac{n!}{n^{n/2} \cdot (n-l)!} \cdot S(n/2, l) \cdot \left(1 - \frac{l}{n}\right)^{n/2} \right)^3 \right).\end{aligned}$$

Here we used that $(x_1 + \cdots + x_m)^3 \leq m^2 \cdot (x_1^3 + \cdots + x_m^3)$. The idea is now to prove that $f(l) = 2^{n/3} \cdot \frac{n!}{n^{n/2} \cdot (n-l)!} \cdot S(n/2, l) \cdot \left(1 - \frac{l}{n}\right)^{n/2}$ decays exponentially in n uniformly in l. For that write $l = c \cdot n/2$ with $c \in (0, 1)$ and use Lemma 3.3 together with the asymptotic formula $n! = n^n \cdot e^{-n+o(n)}$ to obtain

$$\begin{aligned}\frac{\log f(l)}{n} &= \frac{1}{n} \cdot \log \left(2^{n/3} \cdot \frac{n!}{n^{n/2} \cdot (n-l)!} \cdot S(n/2, l) \cdot \left(1 - \frac{l}{n}\right)^{n/2} \right) \\ &= \frac{1}{n} \cdot \log \left(2^{n/3} \cdot \left(1 - \frac{c}{2}\right)^{-n \cdot (1-c/2)} \cdot 2^{-n/2+l} \cdot e^{-cn/2} \cdot e^{g(c) \cdot n/2 + o(n)} \cdot \left(1 - \frac{c}{2}\right)^{n/2} \right) \\ &= \log 2 \cdot \left(\frac{c}{2} - \frac{1}{6} \right) - \left(1 - \frac{c}{2}\right) \log \left(1 - \frac{c}{2}\right) - \frac{c}{2} + \frac{g(c)}{2} + \frac{\log\left(1 - \frac{c}{2}\right)}{2} + o(1) \\ &= \mu(c) + o(1).\end{aligned}$$

Here $g(c) = c + \log \gamma + (\gamma - c) \cdot \log (\gamma - c) - \gamma \cdot \log \gamma$ and γ is given by $\gamma \cdot (1 - e^{-1/\gamma}) = c$. It remains to note that $\mu(c) < -\varepsilon$ for all $c \in (0, 1)$, with some fixed $\varepsilon > 0$. □

Remark 3. It turns out that $\max_c \mu(c)$ is very close to 0, namely, $0 > \max_c \mu(c) > -\frac{1}{500}$.

Theorem 8. *For any $\varepsilon > 0$ the following holds. Suppose two maps $\phi_1, \phi_2 : [n] \to [n]$ are chosen independently according to the uniform measure on the set of maps. Let $p = \sup\{p_{\phi_1}, p_{\phi_2}\}$, then*

$$\lim_{n \to \infty} \mathbb{P}[p \text{ has a block of size at least } n^{\frac{3}{4}+\varepsilon}] = 0.$$

Proof. Consider a graph G on n vertices with edges of two types. We draw an edge of the first type between two vertices i and j if $\phi_1(i) = \phi_1(j)$ and an edge of the second type if $\phi_2(i) = \phi_2(j)$. Note that it is possible that there are edges of both types between i and j. It is clear from the construction that blocks of $p = \sup\{p_{\phi_1}, p_{\phi_2}\}$ are connected components of the graph G. Evidently, as edges of each type form disjoint cliques, if there exists a path in G between i and j then there also exists a simple path between i and j in which types of edges alternate.

Let us now fix two vertices i and j in G and estimate the probability that there exists such an alternating path. For a fixed k, the probability of having an alternating path of length k between i and j is bounded above by $2 \cdot \frac{(n-2)!}{(n-k-1)!} \cdot n^{-k}$. Indeed, $\frac{(n-2)!}{(n-k-1)!}$ is the number of sequences of vertices $\gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_k\}$ between $i = \gamma_0$ and $j = \gamma_k$, and for a fixed sequence γ the probability that it forms an alternating path in G is $2 \cdot (n^{-1})^k$, simply because the probability that two given vertices are connected by an edge of a given type is $\frac{1}{n}$. We thus obtain the following bound:

$$\mathbb{P}[\text{there exists a path between } i \text{ and } j \text{ in } G] \leq 2 \cdot \sum_{k=1}^{n-1} \frac{(n-2)!}{(n-k-1)!} \cdot n^{-k}$$

$$= \frac{2}{n} \cdot \left(1 + \left(1 - \frac{2}{n}\right) + \left(1 - \frac{2}{n}\right) \cdot \left(1 - \frac{3}{n}\right) + \ldots\right)$$

$$\leq \frac{2}{n} \cdot \left(n^{1/2} + n^{1/2} + n^{1/2} \cdot \left(1 - \frac{n^{1/2}}{n}\right)^{n^{1/2}} + n^{1/2} \cdot \left(1 - \frac{n^{1/2}}{n}\right)^{2 \cdot n^{1/2}} + \ldots\right)$$

$$= \frac{2 \cdot n^{1/2}}{n} \cdot \left(1 + 1 + e^{-1} + e^{-2} + \ldots\right) = O\left(n^{-1/2}\right).$$

It follows then immediately that the expected number of pairs $\{i, j\}$ such that i and j are in the same block in p is at most $O(n^{\frac{3}{2}})$. Hence,

$$\mathbb{P}[p \text{ has a block of size at least } n^{\frac{3}{4}+\varepsilon}] \leq \frac{O\left(n^{\frac{3}{2}}\right)}{\left(n^{\frac{3}{4}+\varepsilon}\right)^2} = O(n^{-2\varepsilon}).$$

□

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