Multiplicities of eigenvalues of the Star graph

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Abstract. The Star graph S_n, $n \geqslant 2$, is the Cayley graph on the symmetric group Sym_n generated by the set of transpositions $\{(1\ 2), (1\ 3), \ldots, (1\ n)\}$. We consider the spectrum of the Star graph as the spectrum of its adjacency matrix. It is known that the spectrum of S_n is integral, i.e. it contains only integers. Analytic formulas for multiplicities of eigenvalues $\pm(n-k)$ for $k = 2, 3, 4, 5$ in the Star graph are given in this paper. We also prove that for any eigenvalue of S_n its multiplicity is at least $2^{\frac{1}{2}n \log n(1-o(1))}$.

1 Introduction

In 2002 it was published a survey on integral graphs [4] starting with the question "Which graphs have integral spectra?" by F. Harary and A. J. Schwenk [10]. The problem of characterizing integral graphs seems to be very difficult and so it is wise to restrict ourselves to certain families of graphs. In this paper we are interested in studying Cayley graphs on the symmetric group Sym_n that are also of great interest in computer science as the models of interconnection networks [2, 3, 11, 16].

Let G be a non-trivial group, $S \subseteq G \setminus \{1\}$ and $S = S^{-1} := \{s^{-1} | s \in S\}$. *The Cayley graph* of G denoted by $\Gamma = Cay(G, S)$ is a graph with vertex set G and two vertices a and b are adjacent if $ab^{-1} \in S$. A graph is called *integral* if its adjacency eigenvalues are integers.

The Star graph $S_n = Cay(Sym_n, t)$, $n \geqslant 2$, is the Cayley graph on the symmetric group Sym_n of permutations $\pi = [\pi_1 \pi_2 \ldots \pi_n]$ with the generating set $t = \{(1\ i) \in Sym_n : 2 \leqslant i \leqslant n\}$ of all transpositions $(1\ i)$ swapping the 1st and ith elements of a permutation π.

It is a connected bipartite $(n-1)$–regular graph of order $n!$ and diameter $diam(S_n) = \lfloor \frac{3(n-1)}{2} \rfloor$ [2]. Since this graph is bipartite it does not contain odd cycles but it does contain all even l–cycles where $l = 6, 8, \ldots, n!$ [12] (with the sole exception when $l = 4$). The hamiltonicity of this graph follows from results by V. Kompel'makher and V. Liskovets [14] and by P. J. Slater [18].

We consider the spectrum of the Star graph as the spectrum of its adjacency matrix. In 2009 A. Abdollahi and E. Vatandoost conjectured [1] that the spectrum of S_n is integral, moreover it contains all integers in the range from $-(n-1)$ up to $n-1$ (with the sole exception that when $n \leqslant 3$, zero is not an eigenvalue of S_n). For $n \leqslant 6$ they verified this conjecture numerically using GAP.

In 2012 R. Krakovski and B. Mohar [15] proved that the spectrum of S_n is integral, more precisely, they showed that for $n \geqslant 2$ and for each integer $1 \leqslant k \leqslant n$ the values $\pm(n-k)$ are eigenvalues of the Star graph S_n. Since the Star graph is bipartite, the spectrum of the Star graph is symmetric and $mul(n-k) = mul(-n+k)$ for each integer $1 \leqslant k \leqslant n$ [5]. Let us also note that $\pm(n-1)$ is a simple eigenvalue of S_n. A lower bound on multiplicities of eigenvalues of S_n was also given in [15].

At the same time, G. Chapuy and V. Feray [6] showed another approach to obtain the exact values of multiplicities of eigenvalues of S_n. Their combinatorial approach is based on the Jucys–Murphy elements and the standard Young tableaux. In particular, they gave the following lower bound on multiplicities of eigenvalues of the Star graph:

$$mul(n-k) \geqslant \binom{n-2}{n-k-1}\binom{n-1}{n-k}. \tag{1}$$

In 2015 this approach was used to obtain the multiplicities of eigenvalues of S_n for $n \leqslant 10$ [13].

In this paper we present analytic formulas to calculate multiplicities of eigenvalues of the Star graph.

Theorem 1 *Let $n \geqslant 2$ and for each integer $1 \leqslant k \leqslant n$ the values $\pm(n-k)$ are eigenvalues of the Star graph S_n. The multiplicities $mul(n-k)$ for $k = 2, 3, 4, 5$ of eigenvalues of S_n are given by the following formulas:*

$$mul(n-2) = (n-1)(n-2), \;\; n \geqslant 3; \tag{2}$$

$$mul(n-3) = \frac{(n-1)(n-3)}{2}(n^2-4n+2), \;\; n \geqslant 4; \tag{3}$$

$$mul(n-4) = \frac{(n-1)(n-2)}{3!}(n^4-12n^3+47n^2-62n+12), \;\; n \geqslant 4; \tag{4}$$

$$mul(n-5) = \frac{(n-1)(n-2)}{4!}(n^6-21n^5+169n^4-647n^3+1174n^2-820n+60), n \geqslant 5. \tag{5}$$

From Theorem 1, we immediately have that Chapuy-Feray bound (1) achieved for $k = 2$.

The paper is organized as follows. Section 2 contains three subsections. First we give basic knowledge on the representation theory [17]. Then relationships between this theory and spectra of Cayley graphs are presented. Finally, we show that a formula given by G. Chapuy and V. Feray for multiplicities of eigenvalues of S_n can be rewritten using Hook formula [9]. This new formula is used to prove Theorem 1 in Section 3. We give an improved lower bound on multiplicities of eigenvalues of the Star graph in Section 4.

2 Preliminaries

2.1 Partitions and standard Young tableaux

The symmetric group Sym_n consists of all bijections from $\{1, 2, \ldots, n\}$ to itself using compositions as the multiplication. Any permutation $\pi \in Sym_n$ has the cycle type defined as the unordered list of the sizes of the cycles in the cycle decomposition of π. In this paper we consider a cycle type as a partition.

A *partition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $l \leqslant n$, where the λ_i, $1 \leqslant i \leqslant l$, are weakly decreasing and $\sum_{i=1}^{l} \lambda_i = n$. The *partition function* $P(n)$ represents the number of possible partitions of a natural number n, i.e. the number of distinct ways of representing n as a sum of natural numbers.

A partition λ is presented by its Young tableau. A *standard Young tableau* of shape λ is a filling of the boxes with the elements $\{1, 2, \ldots, n\}$ in such a way that elements increase along rows and columns, all elements are distinct, and each element appears exactly once. The Young tableau of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is the set $[\lambda] = \{(i, j) : 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant \lambda_i\}$. Let us define values $c(m) = i - j$, where $m \in \{1, \ldots, n\}$ and i, j are the ordinate and the abscissa of the box, correspondingly.

We write λ' for the *conjugate partition* of λ defined by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{l'})$, where $l' = \lambda_1$, $\lambda'_j = max\{i : (i, j) \in [\lambda]\}$, $1 \leqslant j \leqslant l'$. Let $(i, j) \in [\lambda]$. The *hook length* h_{ij} is defined by the following formula [9]:

$$h_{ij} = \lambda_i - j + \lambda'_j - i + 1. \tag{6}$$

A *Hook table* H_λ is the table listing the hook length of each box in the standard Young tableau of shape λ.

The definitions above and formula (6) are used in Section 3 to prove Theorem 1.

2.2 Group representations and Cayley graphs

Let G be a group and V be a vector space over the complex numbers and of finite dimension. Let $GL(V)$ stand for the set of all invertible linear transformations of V to itself. Then a *representation* of G on V is a group homomorphism $\rho : G \to GL(V)$. The degree of representation is the dimension of V and is denoted by $dim(V)$. The representation ρ is *irreducible* if V has exactly two subrepresentations, namely the trivial subspace $\{0\}$ and V. Two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are *equivalent* if there exists a bijective linear map $M : V_1 \to V_2$ such that $\rho_2(g)M = M\rho_1(g)$

for all $g \in G$, if $M = 0$, then ρ_1 and ρ_2 are *inequivalent.* A function $f : G \to \mathbb{C}$ is called a *class function* if $f(ghg^{-1}) = f(h)$ for all $g, h \in G$.

Let f be a class function on G and let $\rho : G \to GL(V)$ be a representation of G. Define a *linear map* $\hat{\rho}(f)$ of V into itself by

$$\hat{\rho}(f) = \sum_{g \in G} f(g)\rho(g). \tag{7}$$

Let $\Gamma = Cay(G, f)$ be a Cayley graph with a generating set f, which is identity free and closed under inverses. We view the adjacency matrix A as a linear map $A : \mathbb{C}[G] \to \mathbb{C}[G]$, where $\mathbb{C}[G]$ denote a vector space generated by G over $\mathbb{C}$.

The following theorem connects the spectrum of linear map $\hat{\rho}(f)$ to the spectrum of the Cayley graph.

Theorem 2 [8] *Let $\rho_k : G \to GL(V_k)$ be inequivalent irreducible representations of the finite group G. Let $\hat{\rho}(f)$ be a linear map with the set U_ρ of eigenvalues. Then:*

(1) the set of eigenvalues of the matrix A is presented by $\{\bigcup\limits_{\rho} U_\rho\}$, and

(2) if the eigenvalue μ occurs with multiplicity $mul(\mu)$ in $\hat{\rho}(f)$, then the multiplicity of μ in A is $\sum_k dim(V_k)mul(\mu)$.

This general result was used by G. Chapuy and V. Feray to give the formula for multiplicities of eigenvalues of the Star graph S_n.

2.3 Multiplicities of eigenvalues of the Star graph

We apply the general theory of group representation to the symmetric group Sym_n. The inequivalent irreducible representations of Sym_n are conveniently indexed by the partitions of n. We denote by V_λ the irreducible representation associated with the partition $\lambda \in P(n)$.

The *regular representation* is a representation of a finite group on a subgroup of permutations. It is known [17] that the regular representation of the symmetric group is decomposed into the direct sum of irreducible subrepresentations as follows:

$$\mathbb{C}[Sym_n] = \oplus_{\lambda \in P(n)} dim(V_\lambda)V_\lambda. \tag{8}$$

Group algebra $\mathbb{C}[Sym_n]$ is a vector space of dimension $|Sym_n|$. The elements of Sym_n form a basis for $\mathbb{C}[Sym_n]$. Thus, $\mathbb{C}[Sym_n] = \{c_1s_1 + c_2s_2 + \cdots + c_ks_k : c_i \in \mathbb{C} \text{ for all } i\}$.

The following result shows us how the regular representation of the symmetric group is associated with the multiplicities of eigenvalues of the Star graph.

Theorem 3 [6] *The spectrum of S_n contains only integers. The multiplicity $mul(n-k)$, where $1 \leqslant k \leqslant n-1$, of an integer $(n-k) \in \mathbb{Z}$ is given by:*

$$mul(n-k) = \sum_{\lambda \in P(n)} dim(V_\lambda) I_\lambda(n-k), \tag{9}$$

where $dim(V_\lambda)$ is the dimension of an irreducible representation, $I_\lambda(n-k)$ is the number of standard Young tableaux of shape λ, satisfying $c(n) = n-k$.

Let us note that the dimension of the irreducible representation V_λ of the symmetric group Sym_n corresponding to a partition λ of n is equal to the number of different standard Young tableaux. This number can be calculated by the Hook Formula.

Theorem 4 (Hook Formula) [9] *Let λ be a partition of n. Then,*

$$dim(V_\lambda) = \frac{n!}{\prod_{(i,j)\in[\lambda]} h_{ij}}. \tag{10}$$

Moreover, the number of standard Young tableaux of shape λ such that $c(n) = n-k$ is also calculated by the Hook Formula:

$$I_\lambda(n-k) = \frac{(n-1)!}{\prod_{(i,j)\in[\lambda]} \hat{h_{ij}}}, \tag{11}$$

where $\hat{h_{ij}} = \begin{cases} \lambda_i - j + (\lambda'_j - 1) - i + 1, & j = 1,\ 1 \leqslant i \leqslant l-1; \\ \lambda_i - j + \lambda'_j - i + 1, & 1 < j \leqslant \lambda_i,\ 1 \leqslant i \leqslant l-1. \end{cases}$

From (9), (10) and (11), we immediately get the following lemma.

Lemma 1 *The multiplicity $mul(n-k)$, where $1 \leqslant k \leqslant n$, of the eigenvalues $(n-k) \in \mathbb{Z}$ of the Star graph is given by:*

$$mul(n-k) = \sum_{\lambda \in P(n)} \frac{n!}{\prod_{(i,j)\in[\lambda]} h_{ij}} \cdot \frac{(n-1)!}{\prod_{(i,j)\in[\lambda]} \hat{h_{ij}}}, \tag{12}$$

where $h_{ij} = \lambda_i - j + \lambda'_j - i + 1$ and
$\hat{h_{ij}} = \begin{cases} \lambda_i - j + (\lambda'_j - 1) - i + 1, & j = 1,\ 1 \leqslant i \leqslant l-1; \\ \lambda_i - j + \lambda'_j - i + 1, & 1 < j \leqslant \lambda_i,\ 1 \leqslant i \leqslant l-1. \end{cases}$

3 Proof of Theorem 1

Case 1.

Let us prove formula (2), considering the eigenvalue $(n-2)$ of the Star graph S_n. In this case the standard Young tableaux (SYT) associated with this eigenvalue have the shape $\lambda = (2, 1, 1, \ldots, 1)$ such that the first column contains $(n-1)$ boxes, the remaining box is placed in the second column. Thus, n appears in the topmost box and $c(n) = (n-1)-1 = n-2$ (see Figure 1(a)).

By formula (6), the hook length is the number of boxes that are in the same row i to the right of it plus the number of boxes in the same column j above it, plus one (for the box itself). From the definition of the hook length, we immediately get that for 1-row and 2-column box we have $h_{12} = 1$. The lengths of the hooks for the boxes in the first column are calculated as follows:

$$h_{11} = 1 + (n-2) + 1 = n,$$
$$h_{21} = 0 + (n-3) + 1 = n-2,$$
$$h_{31} = 0 + (n-4) + 1 = n-3,$$
$$\ldots$$
$$h_{(n-1)1} = 0 + 0 + 1 = 1.$$

The Hook table H_λ is presented in Figure 1(b), where each box contains the corresponding hook length. Thus,

$$\prod_{(i,j)\in[\lambda]} h_{ij} = 1 \cdot n(n-2)(n-3)\cdots 1 = n(n-2)!$$

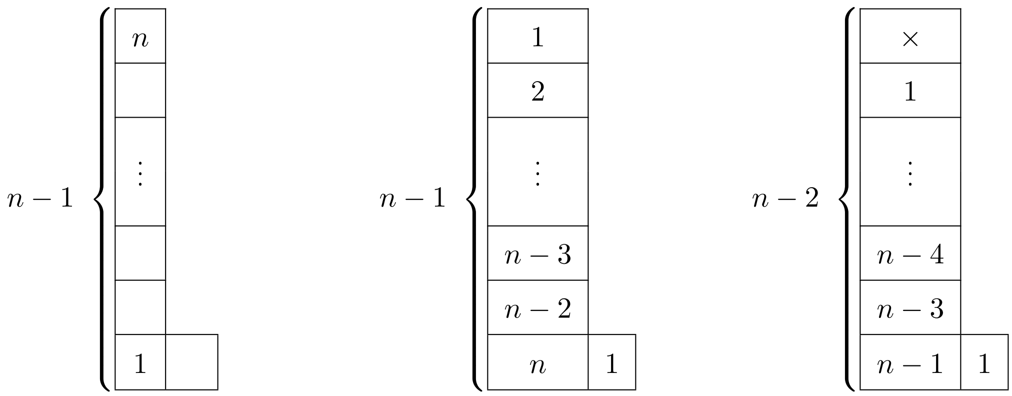
and from (10) we get

$$dim(V_\lambda) = \frac{n!}{n(n-2)!} = \frac{n(n-1)(n-2)!}{n(n-2)!} = (n-1).$$

To obtain $I_\lambda(n-2)$, we need to calculate the hook length of all boxes in SYT without topmost box containing n. This tableau contains $(n-2)$ boxes in the first column, and it is unchanged in the second column. The corresponding Hook table $\bar{H}_\lambda$ for SYT without one topmost box is presented in Figure 1(c).

Hence, from (11) we have:

$$I_\lambda(n-2) = \frac{(n-1)!}{1 \cdot (n-1)(n-3)!} = (n-2),$$



(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 1. $\lambda = (2, 1, 1, \ldots, 1)$

and from (12) we get:

$$mul(n-2) = (n-1)(n-2),\ n \geqslant 3,$$

which gives us (2) in Theorem 1.

Case 2.

Now let us prove (3), considering the eigenvalue $(n-3)$ of the Star graph S_n. In this case we have two standard Young tableaux of shapes $\lambda_1 = (2, 2, 1, \ldots, 1)$ and $\lambda_2 = (3, 1, 1, \ldots, 1)$. Thus, n appears in the topmost box in both of shapes, $c(n) = (n-2) - 1 = n - 3$ (see Figure 2(a) and Figure 3(a), correspondingly).

The Hook tables for standard Young tableaux of shapes $\lambda_1 = (2, 2, 1, \ldots, 1)$ and $\lambda_2 = (3, 1, 1, \ldots, 1)$ are presented in Figure 2(b) and Figure 3(b), correspondingly. Thus, by formulas (6) and (10) for λ_1 we have:

$$\prod_{(i,j)\in[\lambda_1]} h_{ij} = 1 \cdot 2 \cdot (n-1)(n-2)(n-4)\cdots 1 = 2(n-1)(n-2)(n-4)!$$

and

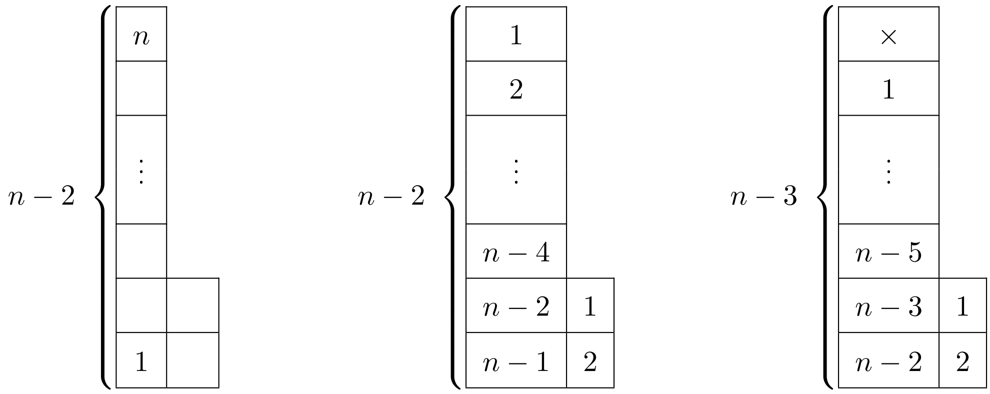
$$dim(V_{\lambda_1}) = \frac{n!}{2(n-1)(n-2)(n-4)!} = \frac{n(n-3)}{2},$$

and for λ_2 we have:

$$\prod_{(i,j)\in[\lambda_2]} h_{ij} = 1 \cdot 2 \cdot n(n-3)(n-4)\cdots 1 = 2n(n-3)!$$

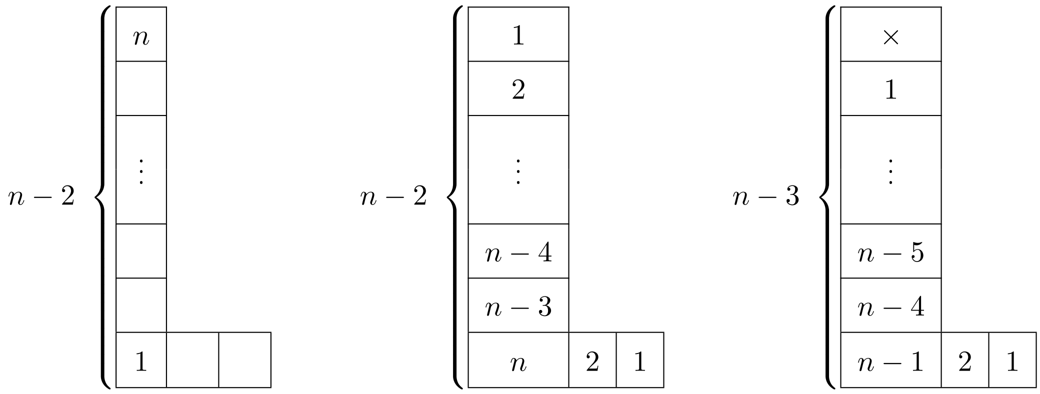
and

$$dim(V_{\lambda_2}) = \frac{n!}{2n(n-3)!} = \frac{(n-1)(n-2)}{2}.$$



(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 2. $\lambda_1 = (2, 2, 1, \ldots, 1)$



(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 3. $\lambda_2 = (3, 1, 1, \ldots, 1)$

Let us find $I_\lambda(n-3)$ for each of the standard Young tableaux of shapes λ_1 and λ_2 without topmost box containing n (see Figure 2(c) and Figure 3(c), correspondingly). By formula (11),

$$I_{\lambda_1}(n-3) = \frac{(n-1)!}{1 \cdot 2 \cdot (n-2)(n-3)(n-5)!} = \frac{(n-1)(n-4)}{2}$$

and

$$I_{\lambda_2}(n-3) = \frac{(n-1)!}{1 \cdot 2 \cdot (n-1)(n-4)!} = \frac{(n-2)(n-3)}{2}.$$

Finally, from (12) we have:

$$mul(n-3) = \frac{n(n-3)}{2} \cdot \frac{(n-1)(n-4)}{2} + \frac{(n-1)(n-2)}{2} \cdot \frac{(n-2)(n-3)}{2} =$$

$$= \frac{(n-1)(n-3)}{2}(n^2 - 4n + 2)$$

for $n \geqslant 4$, which gives us (3) in Theorem 1.

Cases 3 and 4.

Formulas (4) and (5) are proved by similarly reasonings that were used in the previous cases. The standard Young tableaux of the corresponding shapes λ, the Hook tables H_λ and $\bar{H}_\lambda$ are presented in Application.

In the Case 3, there are three SYT of shapes $\lambda_1 = (4, 1, 1, \ldots, 1)$, $\lambda_2 = (3, 2, 1, \ldots, 1)$, $\lambda_3 = (2, 2, 2, 1, \ldots, 1)$ (see Figures 6, 7, 8 in Application). Using data presented in Application for the eigenvalue $(n-4)$ of the Star graph, and by formulas (6), (10) and (11) we get the following results for the considered shapes:

	λ_1	λ_2	λ_3
$dim(V_\lambda)$	$\frac{x_1x_2x_3}{3!}$	$\frac{x_0x_2x_4}{3}$	$\frac{x_0x_1x_5}{3!}$
$I_\lambda(n-4)$	$\frac{x_2x_3x_4}{3!}$	$\frac{x_1x_3x_5}{3}$	$\frac{x_1x_2x_6}{3!}$

where $x_i = (n-i)$ for each $0 \leqslant i \leqslant 6$. Thus, by formula (12) we have:

$$mul(n-4) = \frac{(n-1)(n-2)}{3!}(n^4 - 12n^3 + 47n^2 - 62n + 12),\ n \geqslant 4,$$

which gives us (4) in Theorem 1.

In the Case 4, there are five SYT of shapes $\lambda_1 = (5, 1, 1, \ldots, 1)$, $\lambda_2 = (4, 2, 1, \ldots, 1)$, $\lambda_3 = (3, 3, 1, \ldots, 1)$, $\lambda_4 = (3, 2, 2, 1, \ldots, 1)$, $\lambda_5 = (2, 2, 2, 2, 1, \ldots, 1)$ (see Figures 9 - 13 in Application). Using data presented in Application for the eigenvalue $(n-5)$ of the Star graph, and by formulas (6), (10) and (11) we get the following results for the considered shapes:

	λ_1	λ_2	λ_3	λ_4	λ_5
$dim(V_\lambda)$	$\frac{x_1x_2x_3x_4}{4!}$	$\frac{x_0x_2x_3x_5}{8}$	$\frac{x_0x_1x_4x_5}{12}$	$\frac{x_0x_1x_3x_6}{8}$	$\frac{x_0x_1x_2x_7}{4!}$
$I_\lambda(n-5)$	$\frac{x_2x_3x_4x_5}{4!}$	$\frac{x_1x_3x_4x_6}{8}$	$\frac{x_1x_2x_5x_6}{12}$	$\frac{x_1x_2x_4x_7}{8}$	$\frac{x_1x_2x_3x_8}{4!}$

where $x_i = (n-i)$ for each $0 \leqslant i \leqslant 8$. Hence, by formula (12) we have:

$$mul(n-5) = \frac{(n-1)(n-2)}{4!}(n^6 - 21n^5 + 169n^4 - 647n^3 + 1174n^2 - 820n + 60)$$

for $n \geqslant 5$, which gives us (5) in Theorem 1.

4 Lower bound on multiplicity of eigenvalues of the Star graph

In this section we improve (1) using standard Young tableaux (SYT). Let us put $t = n-k$.

Theorem 5 *In the Star graph S_n for sufficiently large n and for a fixed t the multiplicity $mul(t)$ of eigenvalue t is at least $2^{\frac{1}{2}n\log n(1-o(1))}$.*

Proof Let us consider SYT of size $(m+t)\times m$. Let $i=m+t$ is the number of rows and $j=m$ is the number of columns. Then, n appears in the rightmost and topmost box and $c(n)=i-j=t$. From (9) we have:

$$mul(t)=\sum_{\lambda\in P(n)} dim(V_\lambda)I_\lambda(t). \tag{13}$$

Since $dim(V_\lambda)\geqslant I_\lambda(t)$, then

$$\sum_{\lambda\in P(n)} dim(V_\lambda)I_\lambda(t)\geqslant I_\lambda^2(t). \tag{14}$$

Let us note that in (13) for any t there are SYT of size $(m+t)\times m$ such that they have subtableaux T of size $(m+1)\times m$, where $m\simeq n^{\frac{1}{2}}$. The number of tableaux $I_\lambda(t)$ is at least the number of a standard filling of the tableaux T with the elements $\{1,2,\ldots,(m+1)m\}$ (see Figure 4).

		$\ldots$	$(m+1)m-1$	$(m+1)m$
				$(m+1)m-2$
				$\vdots$
$\vdots$				
2				
1	3	$\ldots$		

Figure 4. The tableaux T

Let us consider the main diagonal and all diagonals above and below the main one in T. Then, $(m+1)m$ appears there in the rightmost and topmost box, and the element $(m+1)m-1$ can appear to the left or below of $(m+1)m$. The same we have for the element $(m+1)m-2$. Thus, we get 2! different STY permuting $(m+1)m-1$ with $(m+1)m-2$ along with the corresponding diagonal. Moreover, the number of subtableaux T is equal to $(1!2!\cdots(m-1)!m!m!(m-1)!\cdots 2!1!)$, when we permute elements along with corresponding diagonal. From (13) and (14), we get:

$$mul(t)\geqslant (1!2!3!\cdots(m-1)!m!)^4=\left(\prod_{i=1}^{m} i!\right)^4\geqslant\left(\prod_{i=\frac{m}{2}}^{m} i!\right)^4\geqslant\left(\left(\frac{m}{2}\right)!\right)^{2m}.$$

We represent Stirling's formula as follows:

$$n!\simeq n^{n(1-o(1))}.$$

Then,

$$\left(\left(\frac{m}{2}\right)!\right)^{2m}\geqslant\left(\left(\frac{m}{2}\right)^{\frac{m}{2}(1-o(1))}\right)^{2m}=\left(\frac{m}{2}\right)^{m^2(1-o(1))}.$$

Since $n\geqslant (m+1)m$, then $m\simeq n^{\frac{1}{2}}$ and we get:

$$\left(\frac{m}{2}\right)^{m^2(1-o(1))}=\left(\frac{n^{\frac{1}{2}}}{2}\right)^{n(1-o(1))}=\left(\frac{1}{2}\right)^{n(1-o(1))}n^{\frac{1}{2}n(1-o(1))}=$$

$$=2^{-n(1-o(1))}2^{\frac{1}{2}n\log n(1-o(1))}=2^{\frac{1}{2}n\log n(1-\frac{2}{\log n})(1-o(1))}=2^{\frac{1}{2}n\log n(1-o(1))}.$$

Finally, we have:

$$mul(t) \geqslant 2^{\frac{1}{2}n\log n(1-o(1))},$$

which completes the proof of Theorem 5. Thus, for all fixed eigenvalues t of the S_n the order of logarithm of multiplicities $mul(t)$ is the same that $n!$. □

5 Conclusion

Formula (12) gives us a method to get analytic formulas for calculating the multiplicities of eigenvalues of the Star graph S_n for any n. To realize this, a computer program was written in Python. The program allows to calculate exact values of multiplicities up to $n = 50$. The exact values of multiplicities of eigenvalues of S_n for $2 \leqslant n \leqslant 23$ are presented in Figure 5, where the abscissa corresponds to the units eigenvalues and the ordinate with a logarithmic scale corresponds to multiplicities. This graphic shows us a polynomial growth of multiplicities of most eigenvalues and extraexponential growth of multiplicities of least eigenvalues. In particular, the multiplicity of 0 is a local maximum of S_n for $n = 4 \div 7, 13 \div 18$, and local minimum of S_n for $n = 8 \div 12, 19 \div 23$. This alternation is of great interest, and the question of its causes is still open.

In general, the obtained results on the distribution of eigenvalue multiplicities for the Star graphs quite differ from the known facts for integral Cayley graphs. For example, the Hypercube graphs have the normal distribution of eigenvalue multiplicities [5, 7]. In the case of the Star graphs we have an interesting example of unknown distribution.

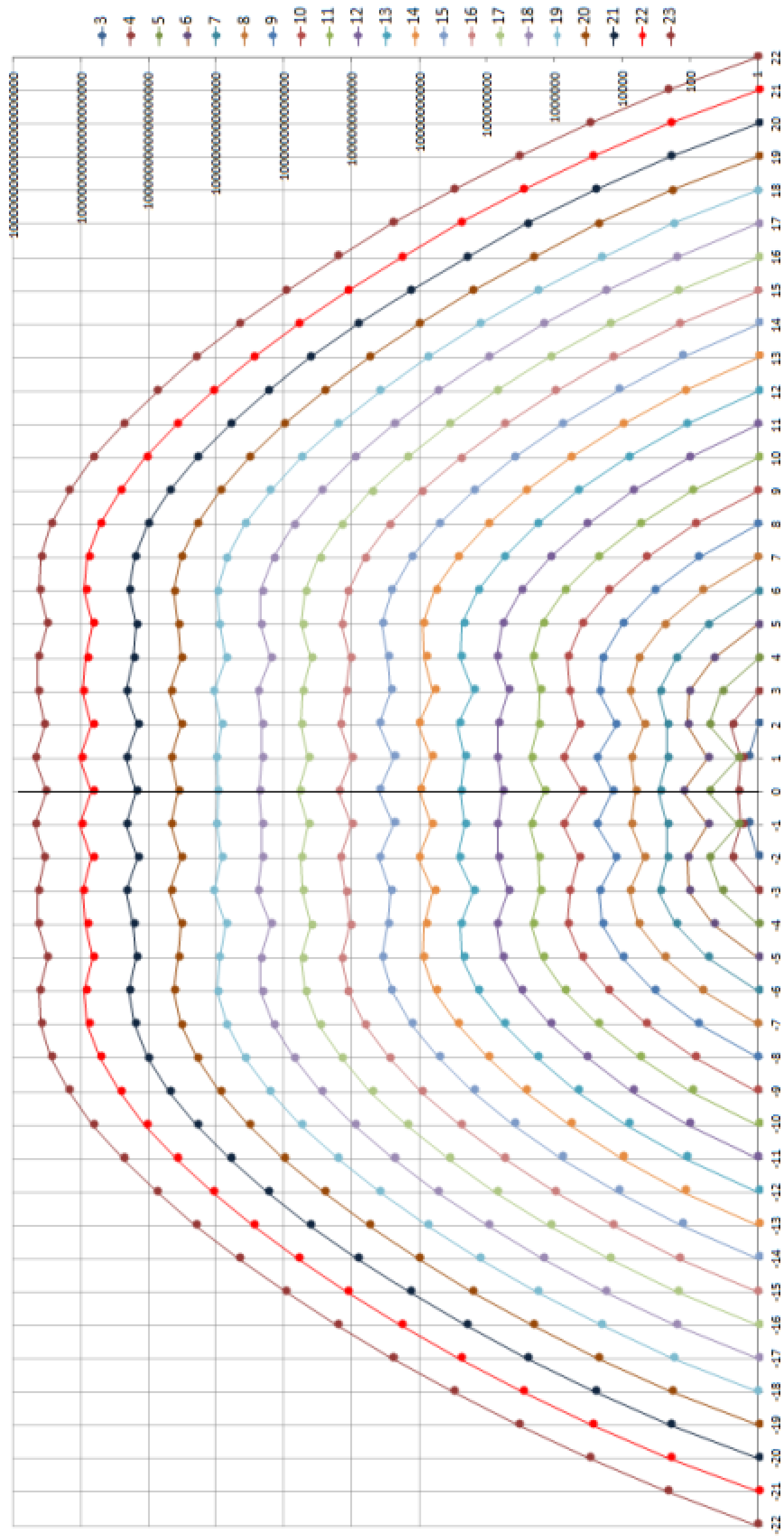


Figure 5. Graphic representations of multiplicities of eigenvalues of S_n for $2 \leqslant n \leqslant 23$

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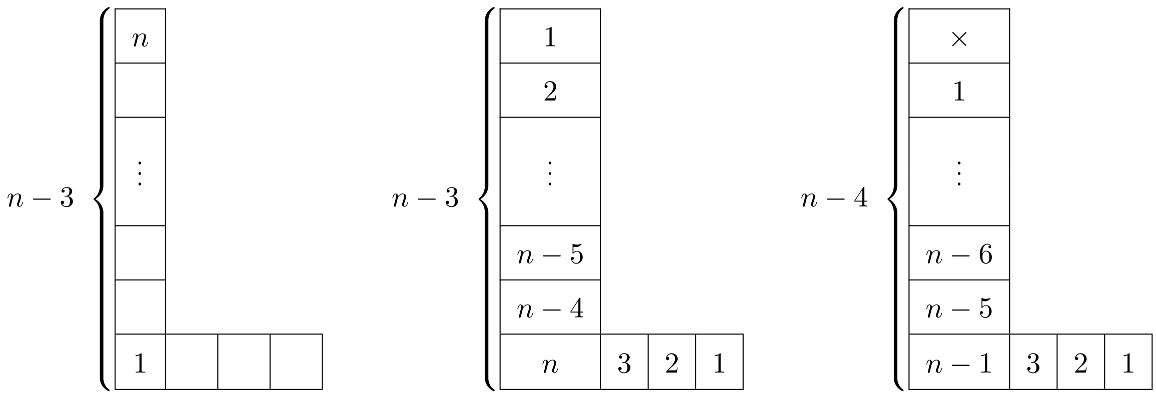
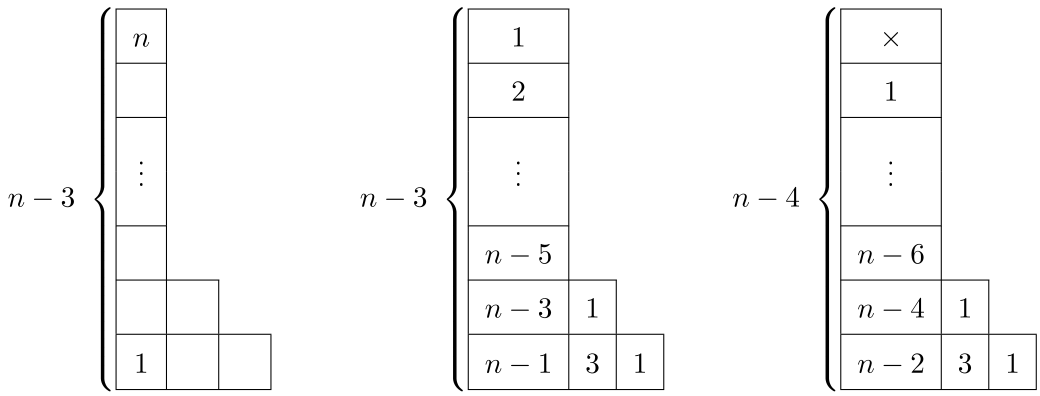
Application



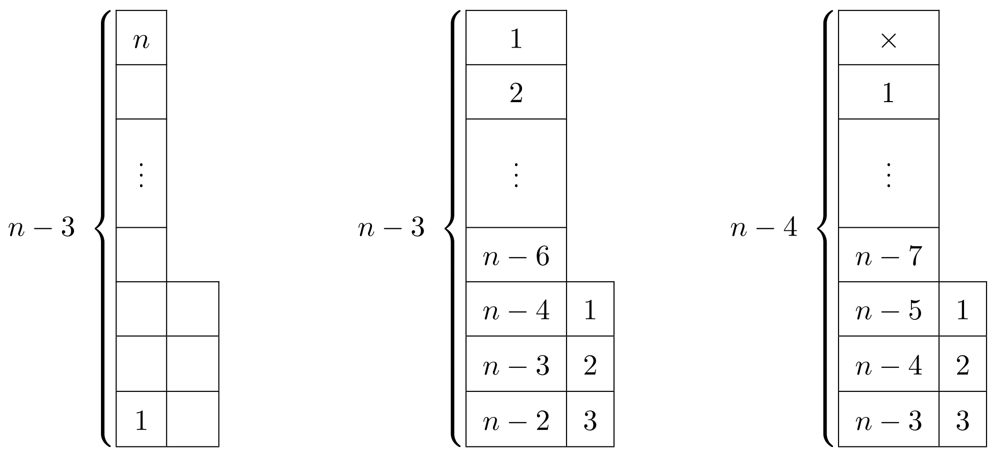
(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 6. $\lambda_1 = (4, 1, 1, \ldots, 1)$



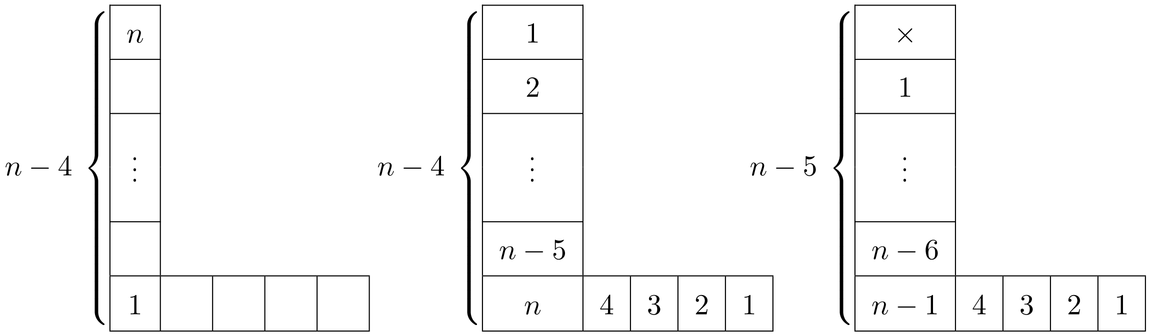
(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 7. $\lambda_2 = (3, 2, 1, \ldots, 1)$



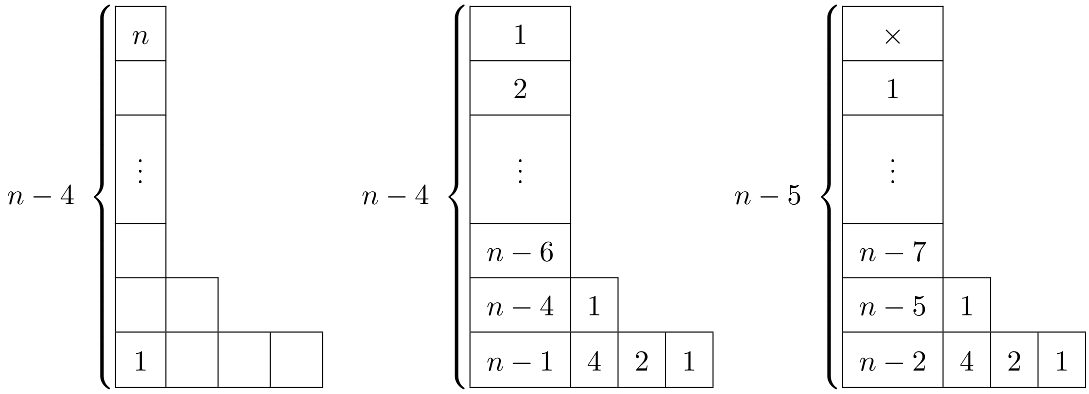
(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 8. $\lambda_3 = (2, 2, 2, 1, \ldots, 1)$



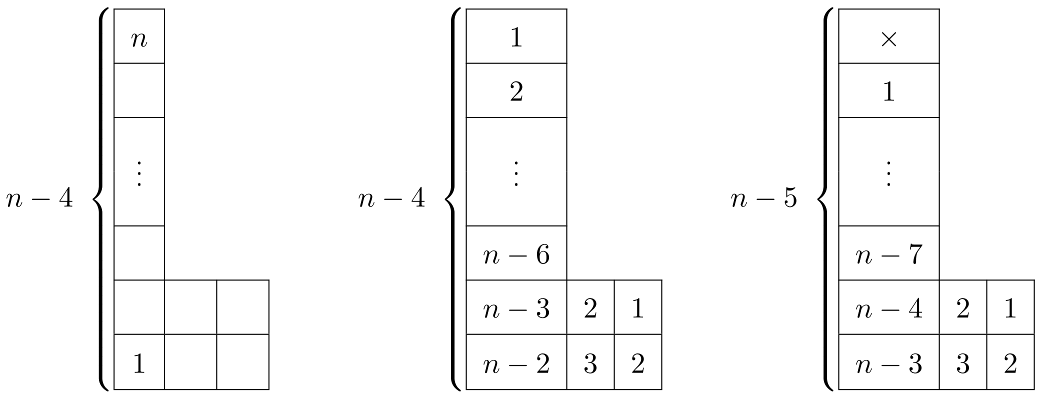
(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 9. $\lambda_1 = (5, 1, 1, \dots, 1)$



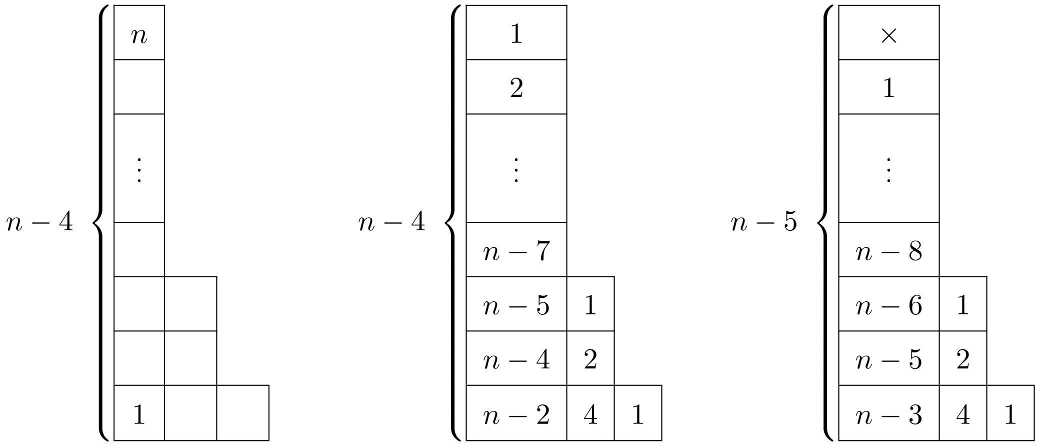
(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 10. $\lambda_2 = (4, 2, 1, \dots, 1)$



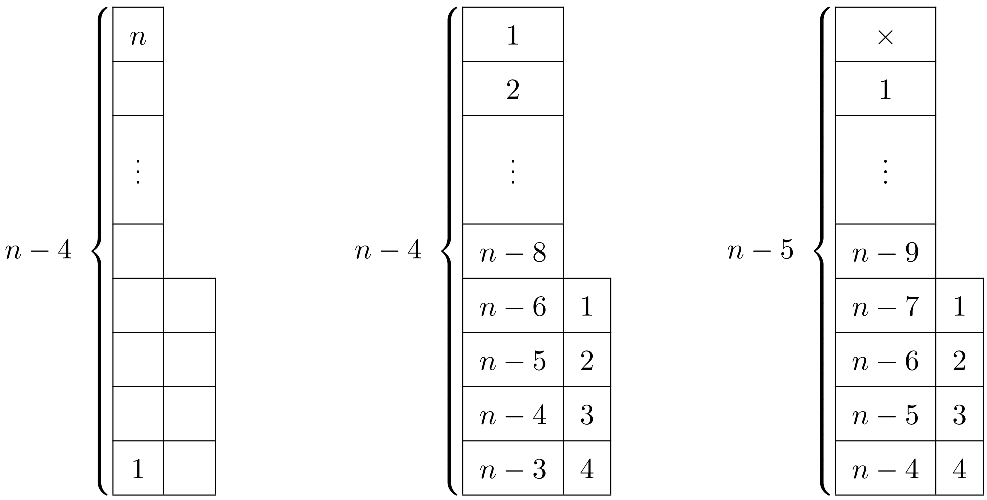
(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 11. $\lambda_3 = (3, 3, 1, \dots, 1)$



(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 12. $\lambda_4 = (3, 2, 2, 1, \ldots, 1)$



(a) SYT (b) H_λ (c) $\bar{H}_\lambda$

Figure 13. $\lambda_5 = (2, 2, 2, 2, 1, \ldots, 1)$