Combinatorics of the Lipschitz polytope

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Abstract

Let ρ be a metric on the set $X = \{1, 2, \ldots, n + 1\}$. Consider the n-dimensional polytope of functions $f : X \to \mathbb{R}$, which satisfy the conditions $f(n + 1) = 0$, $|f(x) - f(y)| \leqslant \rho(x, y)$. The question on classifying metrics depending on the combinatorics of this polytope has been recently posed by A. M. Vershik [1]. We prove that for any "generic" metric the number of $(n - m)$-dimensional faces, $0 \leqslant m \leqslant n$, equals $\binom{n+m}{m,m,n-m} = (n + m)!/m!m!(n - m)!$. This fact is intimately related to regular triangulations of the root polytope (convex hull of the roots of A_n root system). Also we get two-sided estimates for the logarithm of the number of Vershik classes of metrics: $n^3 \log n$ from above and n^2 from below.

1   Introduction

Let (X, ρ) be a finite metric space with $|X| = n + 1$. A metric ρ will be called *strict*, if $\rho(x, z) < \rho(x, y) + \rho(y, z)$ for $y \in X \setminus \{x, z\}$. On the space of functions from X

$$\tilde{F} := \{f : X \to \mathbb{R}\}$$

we define a map, which maps every function to its Lipcshitz constant:

$$\|f\| := \max_{x,y \in X} \frac{f(y) - f(x)}{\rho(x, y)}.$$

It is a seminorm, becoming a norm if all functions differing by a constant are identified — or, equivalently, if the value is fixed at one point:

$$F := \left\{f \in \tilde{F} \mid f(x_0) = 0\right\}, x_0 \in X.$$

The closed unit ball of the norm $\| \cdot \|$ on the space F is a convex n-dimensional polytope, which will be denoted by $\mathrm{LIP}(X)$. We say that a function f on X is 1-Lipschitz iff

$f \in \mathrm{LIP}(X)$, that is, $|f(x) - f(y)| \leqslant \rho(x, y)$ for all $x, y \in X$. We will interpret the dual normed space F^* as a space of signed measures μ on the metric space (X, ρ) with the total measure of 0, the pairing $\langle \mu, f \rangle$ of a function f and a signed measure μ is $\int f d\mu$ (the value doesn't change when we add a constant to f, thus it is well defined.) The delta-measure at a point $x \in X$ will be denoted by δ_x, and then the signed measure $\mu \in F^*$ has the following form:

$$\mu = \sum_{x \in X} c_x \delta_x, \ \sum c_x = 0;$$

$$\langle \mu, f \rangle = \sum_{x \in X} c_x f(x), \ \|\mu\| = \max_{f \in \mathrm{LIP}(X)} \langle \mu, f \rangle.$$

This norm of a signed measure $\mu \in F^*$ is called *Kantorovich–Rubinstein norm*. It is equal to the Kantorovich optimal transportation distance between the measures μ_+ and μ_- coming from the Hahn decomposition of $\mu = \mu_+ - \mu_-$.

The convex hull of the set of points of the form $e_{x,y} := \frac{\delta(x) - \delta(y)}{\rho(x,y)}, x, y \in X$ serves as a dual polytope to $\mathrm{LIP}(X)$, i.e., the unit ball $\mathrm{KR}(X)$ of the Kantorovich–Rubinstein space F^*: indeed, the norm of a function f, by definition, satisfies

$$\|f\| = \sup_{x,y \in X} \langle e_{x,y}, f \rangle.$$

A. M. Vershik posed a question about the combinatorial structure of polytopes $\mathrm{KR}(X)$ (equivalently, about combinatorial structure of $\mathrm{LIP}(X)$.)

Note that the same family of polytopes (for non symmetric metrics, but it is not crucial) appears in the tropical mathematics [12, 14] as the cells of tropical convex polytopes and as the tropical polytopes which are convex in the usual sense; and also as "alcoved polytopes" [13].

For the special metric $\rho(x, y) =: \mathbf{1}(x, y) = 1$ at $x \neq y$, the polytope $\mathrm{KR}(X)$ is called *root polytope*, as its vertices are precisely the roots of the A_n root system. This polytope and its triangulations were studied in [9]. It is helpful in the study of the Lipschitz polytopes for general metrics, cf. Section 6.

Definition 1. We say that a metric ρ is *generic*, if ρ is strict and the polytope $\mathrm{KR}(X)$ is simplicial (or, equivalently, the polytope $\mathrm{LIP}(X)$ is simple).

We compute the f-vectors of this polytope $\mathrm{KR}(X)$ for a generic metric. Furthermore, we estimate from both sides number of types of metrics on the set $\{1, \ldots, n+1\}$, classified by the combinatorial type of the (naturally labelled) polytope KR.

Apparently, in the generic situation the f-vector of the polytope LIP does not depend on the metric:

Theorem 1. *Let X, $|X| = n + 1$ be a metric space with generic metric ρ. Then, for $0 \leqslant m \leqslant n$, the number of $(n - m)$-dimensional faces of the polytope $\mathrm{LIP}(X)$ is equal to $\binom{n+m}{m,m,n-m} = \frac{(n+m)!}{m!m!(n-m)!}$.*

Associate an oriented edge from x to y to a point $e_{x,y}$. Thereby any face α (of arbitrary dimension) of the polytope $\mathrm{KR}(X)$ is associated to the graph $D(\alpha)$ on the vertex set X with edges corresponding to signed measures $e_{x,y}$ lying on the face α. Let $\tilde{D}(\alpha)$ denote the same graph $D(\alpha)$ with orientation forbidden.

Definition 2. The collection of graphs of the form $D(\alpha)$ is called a *combinatorial structure* of a dual pair of polytopes $\mathrm{LIP}(X)$, $\mathrm{KR}(X)$. An oriented (unoriented, respectively) graph G on the vertex set X is called *admissible*, if all its edges belong to some graph of the form $D(\alpha)$ ($\tilde{D}(\alpha)$, respectively.) Two metrics ρ_1, ρ_2 on the same set X are called *Lipschitz combinatorially equivalent*, if the combinatorial structures of respective polytopes coincide.

Example 1. *Consider the case $\rho = \mathbf{1}$. Then $\mathrm{KR}((X, \mathbf{1})) := \mathrm{Root}(X)$ is a root polytope, which is a convex hull of all $e_{x,y}$. A face α of $\mathrm{Root}(X)$ is determined by a function f on X which belongs to a unit sphere of $\mathrm{LIP}(X)$, that is, $\max(f) - \min(f) = 1$. The graph $D(\alpha)$ contains all edges from the set $f^{-1}(\max(f))$ to the set $f^{-1}(\min(f))$.*

The following theorem partially answers the question of A. M. Vershik [1].

Theorem 2. *Suppose $|X| = n + 1$. The number $V(n)$, $V_g(n)$ of types of Lipschitz equivalence of, respectively, all metrics and generic metrics on the set X satisfies the inequality*

$$c_1 n^2 \leqslant \log V_g(n) \leqslant \log V(n) \leqslant c_2 n^3 \log(n+1),$$

for some positive absolute constants c_1, c_2.

Note that in [15] an algorithm for enumerating tropical types of the polytropes (tropical type is a refinement of a combinatorial type) was proposed.

2 Combinatorial description of faces

The combinatorial properties of the polytopes $\mathrm{KR}(X)$ have been considered before [7, 8], and some results of this section have been obtained there. But these results are not enough for the purpose of this paper, this is why we formulate and prove here everything that we use.

A face α is the intersection of the polytope $\mathrm{KR}(X)$ and some support hyperplane defined by the equation $\langle \mu, f_0 \rangle = 1$ for appropriate function $f_0 \in \mathrm{LIP}(X)$. The graph $D(\alpha)$ may be described in terms of f_0 as follows: edge from x to y exists iff $f_0(x) - f_0(y) = \rho(x, y)$.

Lemma 1. *Dimension of a face α equals $n - c$, where c is the number of connected components of $\tilde{D}(\alpha)$.*

Proof. Assume that $X_1 \subset X$. Then the signed measures $e_{x,y}$, $x, y \in X_1$, generate a linear space of dimension $|X_1| - 1$. Next, if G is a connected non-oriented graph on the ground set X_1, the signed measures $e_{x,y}$ for $(x, y) \in E(G)$ generate this linear space (since $e_{x,z}$ is a linear combination of $e_{x,y}$ and $e_{y,z}$). Thus the dimension of the linear span of points $e_{x,y}$, which belong to the face α, equals $n + 1 - c$, where c is the number of connected components in $\tilde{D}(\alpha)$. Dimension of affine span is lesser by 1, since affine span of a face does not contain the origin. $\qquad \square$

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Next theorem describes when a given set of signed measures belongs to the same face of the polytope $\mathrm{KR}(X)$.

Theorem 3. *Let $G = (X, E)$ be an oriented graph on the ground set of vertices X. The following four conditions are equivalent:*

(i) the graph G is admissible, i.e., there exists a facet α of the polytope $\mathrm{KR}(X)$ such that $E \subset D(\alpha)$;

(ii) there exists a 1-Lipschitz function f on X such that $f(x) - f(y) = \rho(x, y)$ whenever $(x, y) \in E$;

(iii) for any array of oriented edges $(x_i, y_i) \in E$, $i = 1, \ldots, k$, the inequality holds (here $y_{k+1} = y_1$):

$$\sum_{i=1}^{k} \rho(x_i, y_i) \leqslant \sum_{i=1}^{k} \rho(x_i, y_{i+1}); \tag{1}$$

(iv) inequality (1) holds under additional assumption that all points $x_1, x_2, \ldots, x_k$ are distinct and all points $y_1, y_2, \ldots, y_k$ are distinct.

Proof. A support hyperplane to the unit ball $\mathrm{KR}(X)$ of the space F^* is given by the equation $\langle \mu, f \rangle = 1$, where f is a function from the unit sphere $\mathrm{LIP}(X)$ of the space F. It yields that (i) and (ii) are equivalent. Clearly (iii) implies (iv). If (ii) holds, then (1) follows from

$$\sum_{i=1}^{k} \rho(x_i, y_i) = \sum_{i=1}^{k} (f(x_i) - f(y_i)) = \sum_{i=1}^{k} (f(x_i) - f(y_{i+1})) \leqslant \sum_{i=1}^{k} \rho(x_i, y_{i+1}).$$

It remains to prove (ii) assuming (iv).

Existence of a necessary function f may be rephrased as follows: the subspaces

$$\{f \in F : f(y) - f(x) \leqslant \rho(x, y)\}, (x, y) \in X \times X;$$
$$\{f \in F : f(x) - f(y) \geqslant \rho(x, y)\}, (x, y) \in E,$$

must have non-empty intersection. Since F is n-dimensional, by Helly theorem it suffices to prove that any subfamily of at most $n+1$ subspaces has non-empty intersection. Assume the contrary and consider the least counterexample: at first, by the number $n+1$ of points in X, next, by the number $m \leqslant n+1$ of subspaces with empty intersection. Each subspace is defined by $f(y) - f(x) \leqslant \pm\rho(x, y)$, sign minus is possible if $(x, y) \in E$. Call x a starting point and y an endpoint. If some point x does not serve neither as a starting point nor as an endpoint for none of our subspaces, we have a counterexample with $X \setminus \{x\}$ instead of X. The same holds if x serves only as a staring point or only as an endpoint: a function from $X \setminus \{x\}$ may be extended to x so that inequalities containing $f(x)$ become true. Thus $m = n + 1$ and each point $x \in X$ is an endpoint for exactly one subspace and a starting point for exactly one subspace. A map, which sends each starting point to the corresponding endpoint, is a permutation of X. Let $z_1 \ldots z_s z_1$ be one of its cycles, than f should satisfy inequalities of the form

$$f(z_{i+1}) - f(z_i) \leqslant \varepsilon_i \rho(z_i, z_{i+1}), i = 1, \ldots, s, \varepsilon_i \in \{-1, +1\},$$

(as usual, agree that $z_{s+1} = z_1$). Such a function exists if and only if $\sum \varepsilon_i \rho(z_i, z_{i+1}) \geqslant 0$. Let A be a set of indices i for which $\varepsilon_i = -1$, B be a set of other indices. Then $(z_i, z_{i+1}) \in E$ for $i \in A$. Let $w(i)$, for $i \in A$, denote the (first) index preceding i in a cycle such that $w(i) \in A$. Function w is a cyclic permutation of A. Condition (iv) yields that

$$\sum_{i \in A} \rho(z_i, z_{i+1}) \leqslant \sum_{i \in A} \rho(z_i, z_{w(i)+1}) \leqslant \sum_{i \in B} \rho(z_i, z_{i+1}).$$

(The last inequality follows from several triangle inequalities which are summed up.). This is what we need. $\qquad \square$

Corollary 1. *1) All signed measures $e_{x,y}$ lie on the boundary of the polytope $\mathrm{KR}(X)$.*

2) If a graph $G(\alpha)$ contains edges (x, y), (y, z), then $\rho(x, z) = \rho(x, y) + \rho(y, z)$. In particular, if ρ is a strong metric, then for any vertex of the graph $G(\alpha)$ either indegree or outdegree equals 0. In this case all signed measures $e_{x,y}$ are vertices of the polytope $\mathrm{KR}(X)$.

3) A strong metric ρ is generic if and only if $\tilde{D}(\alpha)$ is a forest for any face α of the polytope $\mathrm{KR}(X)$.

4) Assume that a metric ρ is strict and for all mutually distinct points $x_1, \ldots, x_k, y_1, \ldots, y_k$, $y_{k+1} = x_1$, inequality

$$\sum \rho(x_i, y_i) \neq \sum \rho(x_i, y_{i+1}),$$

takes place. Then the metric ρ is generic.

Proof. 1) Graph with one edge satisfies (iv).

2) Set $x_1 = x, x_2 = y_1 = y, y_2 = z$ in (1), we get $\rho(x, z) \geqslant \rho(x, y) + \rho(y, z)$, thus equality takes place. If ρ is a strict metric, then the function $f(z) = (\rho(y, z) - \rho(x, z))/2 \in \mathrm{LIP}(X)$ satisfies equality $f(a) = f(b) + \rho(a, b)$ only for $a = x$, $b = y$. Thus corresponding support plane has unique common point $e_{x,y}$ with the ploytope $\mathrm{KR}(X)$. This yields that $e_{x,y}$ is a vertex of $\mathrm{KR}(X)$.

3) Strict metric ρ is generic iff the polytope $\mathrm{KR}(X)$ is simplicial. That is, each face α of dimension $k = \dim \alpha$ contains exactly $k + 1$ vertices of $\mathrm{KR}(X)$. By Lemma 1 we have $k = n - c$, where c is a number of connected components of $\tilde{D}(\alpha)$. By p. 2) the number of vertices of $\mathrm{KR}(X)$ which belong to α equals the number of edges of the graph $\tilde{D}(\alpha)$. So, in terms of the graph $\tilde{D}(\alpha)$ condition is the following: sum of the number $k + 1 = n - c + 1$ of edges and the number of connected components c should be equal to the number of vertices $n + 1$. Such graphs are exactly forests.

4) Consider a facet α of the polytope $\mathrm{KR}(X)$. We have to check that α has exactly n vertices, i.e., that the graph $D(\alpha)$ has exactly n edges. The graph $\tilde{D}(\alpha)$ is connected by Lemma 1. Thus it has at least n edges. each vertex of the graph $D(\alpha)$ has indegree or outdegree 0, so, any cycle in $\tilde{D}(\alpha)$ is alternating: $y_1 x_1 y_2 x_2 \ldots y_k x_k$, $(x_i, y_i), (x_i, y_{i+1}) \in E(D(\alpha))$. Using (1) twice we get $\sum \rho(x_i, y_i) \leqslant \sum \rho(x_i, y_{i+1})$ and $\sum \rho(x_i, y_{i+1}) \leqslant \sum \rho(x_i, y_i)$, thus equality takes place. This contradicts to our assumption. Therefore there are no cycles and the non-oriented graph $\tilde{D}(\alpha)$ is a tree, it has exactly n edges, as desired. $\qquad \square$

Condition (iv) of Theorem 3 may be weakened in the case of facets. Namely, we have

Theorem 4. *Let T be a tree on the ground set of vertices X. Orient T so that the indegree or the outdegree of each vertex is 0 (there are two ways to do it.) Obtained oriented graph T_d is contained in some $D(\alpha)$ for a certain facet α of the polytope $\mathrm{KR}(X)$ if and only if inequality (1) takes place for any simple path $y_1 x_1 y_2 x_2 \ldots y_k x_k$ in the (non-oriented) tree T.*

Proof. This condition is necessary by Theorem 3. Let us prove that it is sufficient. There exists the unique (up to additive constant) function f on X such that $f(x) - f(y) = \rho(x, y)$ for each of n directed edges (x, y) of the graph T_d. Our goal is to prove that it satisfies $f(y) - f(x) \leqslant \rho(x, y)$ for all vertices $x, y \in X$. The proof goes by induction in the length (number of edges) of the path P from y to x in T. For paths of length 1 it follows from the definition of f. Assume that we established this for paths shorter than between x and y. Then $f(x) = f(z) \pm \rho(x, z)$, where z precedes x in P. On the other hand, $f(y) - f(z) \leqslant \rho(y, z)$ by induction proposition. Thus

$$f(y) - f(x) = f(y) - f(z) - (f(x) - f(z)) \leqslant \rho(y, z) \mp \rho(x, z). \tag{2}$$

Since $\rho(y, z) - \rho(x, z) \leqslant \rho(x, y)$ by triangle inequality, we get desired inequality $f(y) - f(x) \leqslant \rho(x, y)$ if the sign in (2) is negative. It remains to consider the case $f(x) = f(z) - \rho(x, z)$. Analogously we may suppose that $f(y) = f(x_1) + \rho(y, x_1)$, where x_1 follows after y in P. So, P has even number of vertices, $P = y_1 x_1 \ldots y_k x_k$, $y = y_1$, $x = x_k$,

$$f(y) - f(x) = \sum_{i=1}^{k} \rho(x_i, y_i) - \sum_{i=1}^{k-1} \rho(x_i, y_{i+1}) \leqslant \rho(x_k, y_1) = \rho(x, y)$$

due to (1). $\qquad\square$

Now we give a criterion that ρ is generic.

Theorem 5. *1) For all distinct points $x_1, \ldots, x_k$ and distinct points $y_1, \ldots, y_k$ in X the set of directed edges $(x_i, y_i), 1 \leqslant i \leqslant k$ is admissible if and only if*

$$\sum_{i=1}^{k} \rho(x_i, y_i) = \min_{\pi} \sum_{i=1}^{k} \rho(x_i, y_{\pi(i)}), \tag{3}$$

where minimum is taken over all permutations π of the set $\{1, \ldots, k\}$.

2) A strict metric ρ is generic if and only if for any $2k$ distinct points $x_1, \ldots, x_k$, $y_1, \ldots, y_k$ in X the minimum of a sum in right hand side of equation (3) is attained for unique permutation π.

Proof. 1) Let $C_1, C_2, \ldots$ be non-empty disjoint subsets of $\{1, \ldots, k\}$ which are supports of the cycles of π. If the set of edges (x_i, y_i) is admissible, then for each C_j we have inequality

$$\sum_{i \in C_j} \rho(x_i, y_i) \leqslant \sum_{i \in C_j} \rho(x_i, y_{\pi(i)})$$

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by (1). Summing up we get

$$\sum_{i=1}^{k} \rho(x_i, y_i) \leqslant \sum_{i=1}^{k} \rho(x_i, y_{\pi(i)}),$$

this yields (3) as π is arbitrary. Conversely, if the set of edges (x_i, y_i) is not admissible, condition (iv) of Theorem 3 means that for a certain cyclic permutation of some subset $C \subset \{1, \ldots, k\}$ the value of right hand side of (3) is less than for the identical permutation.

2) If a strong metric ρ is not generic, than some admissible graph $\tilde{D}(\alpha)$ contains a cycle. By p.2 of Corollary 1, each vertex of $D(\alpha)$ has indegree 0 or outdegree 0, thus the vertices of this cycle alternate and we may denote it $y_1 x_1 \ldots y_k x_k y_1$, $(x_i, y_i), (x_i, y_{i+1}) \in D(\alpha)$. It means that the minimum in (3) is obtained both for the identical permutation π and for $\pi(i) = i + 1 \pmod{k}$.

Assume now that ρ is generic strict metric, but the minimum in (3) is obtained for two different permutations. Changing notations and considering a subset on which one of these two permutations is a cyclic shift of another, we may consider the case when one of two permutations is identical and another is a cyclic shift $\pi(i) = i + 1 \pmod{k}$. Let us show that a union of edges of the cycle $y_1 x_1 \ldots y_k x_k y_1$ is admissible, by p.4 of Corollary 1 this contradicts to our assumption that ρ is generic. We check condition (iv). Choose few disjoint edges from the cycle. They belong to some tree obtained from the cycle by removing one edge. Thus it suffices to check that such a tree is admissible. This may be checked by Theorem 4. Indeed, the condition of Theorem for any path in this tree follows from the minimality of one or another permutation. $\square$

3   Stars and trees

In this section we suppose that ρ is a generic metric.

Definition 3. A directed graph is called a *star*, if there are edges coming from one vertex to all the others, and there are no other edges. *Constellation* is a directed graph in which all weak connected components are stars.

Lemma 2. *Let $V = \{v_1, \ldots, v_k\} \subset X$ be a set consisting of k points of a metric space X, $p_1, \ldots, p_k$ are non-negative integers and*

$$k + \sum_{i=1}^{k} p_i = n + 1.$$

Then there exists a unique admissible directed graph D^* such that outdegrees of the vertices v_i are equal to p_i for $i = 1, \ldots, k$ (and their indegrees are equal to 0); indegrees of all vertices in $X \setminus V$ are equal to 1 (and outdegrees are equal to 0). Also, the graph D^* is a constellation and it minimizes the value of the functional

$$F(D) = \sum_{(a,b) \in D} \rho(a, b)$$

on the set of all graphs with described degrees.

Proof. It is clear that any graph with described degrees is a constellation.

In order to prove existence we consider the constellation D^*, which minimizes F. Choose an array of edges as in p. (iv) of Theorem 3. The graph D^* does not contain edges of the form (x_{i+1}, y_i), since degrees of endpoints are equal to 1. Thus $D^* \setminus \{(x_i, y_i)\} \cup \{(x_{i+1}, y_i)\}$ is a constellation again, and the value of F is not less than for D^*. It yields condition (iv) of Theorem 3, so, D^* is indeed admissible.

Now assume that there exists yet another constellation D' with the same degrees. Choose the vertices $x_1 \in V, y_1 \in X \setminus V$ so that $(x_1, y_1) \in D' \setminus D^*$. Since degree of y_1 in both D^* and D' equals 1, there exist $x_2 \in V$ such that $(x_2, y_1) \in D^* \setminus D'$. Next, degree of x_2 is the same in D^* and D'. Thus there exists a vertex $y_2 \in X \setminus V$ such that $(x_2, y_2) \in D' \setminus D^*$. This process continues until $x_m = x_1$ for some m . We got a cycle (strictly speaking, disconnected orientation of a non-directed cycle), all even edges of the cycle belong to the first constellation and all odd edges to another. Since our two constellations are admissible, both sets of odd and even edges minimize the sum in (3). It is impossible for a generic metric by Theorem 5. □

The following statement is straightforward.

Claim 1. *Let T be a tree on the ground set of vertices X, vertices of T are properly colored in black and white colors and $u \in X$ is a white vertex. Consider all edges xy of the tree T such that x is white and the shortest path from u to x does not contain the edge xy. Such edges form a constellation which we denote by $H(T, u)$. Denote by $\Phi_u(T)$ the sum $\sum \rho(x, y)$ of lengths of all edges in $H(T, u)$. Let $P = y_1 x_1 y_2 x_2 \ldots y_k x_k$ be a simple path in the tree T, in which the vertices x_i are white Denote by T' the tree which is obtained from T by the change of the edge $x_k y_k$ to $x_k y_1$. Also denote by w the vertex of P which is closest to u. Then*

$$\Phi_u(T') - \Phi_u(T) = \begin{cases} \rho(x_k, y_1) - \rho(y_1, x_1) + \rho(x_1, y_2) - \cdots - \rho(y_k, x_k), & \text{if } w = x_k \\ 0, & \text{else.} \end{cases}$$

Theorem 6. *Let $X = \{v_1, \ldots, v_{n+1}\}$, non-negative integers $p_1, \ldots, p_{n+1}$ sum up to $\sum p_i = n$. Then there exists unique admissible graph such that outdegree of v_i equals p_i for all $i = 1, \ldots, n+1$.*

If we color the vertices v_i with $p_i > 0$ in white and other vertices in black, then this graph is a directed (from white to black) tree, and for each white vertex u it minimizes the functional Φ_u defined in the statement 1.

Proof. P.4 of Corollary 1 yields that any admissible graph is a (somehow oriented) forest. Thus admissible graph with n edges is a tree.

Let us prove uniqueness of such a tree T. For any white vertex u the constellation $H(T, u)$ is admissible. Degrees of white vertices in $H(T, u)$ depend on u, but not on T. Lemma 2 implies that it is unique and minimizes Φ_u. The tree T is a union of all such constellations $H(T, u)$, thus it is at most unique.

It remains to prove the existence. Consider the tree T with given outdegrees of white vertices, for which the sum of functionals Φ_u (u runs over white vertices) is minimal possible. Let us claim that it is unique by verifying conditions of Theorem 4. By Claim 1, if some path $y_1 \ldots x_{2k}$, where x_i are white, obeys condition of Theorem 4, then for the tree T' each functional Φ_{v_i} takes a value lesser or equal than for T, and some functionals take strictly lesser value. This contradicts the minimality assumption. $\qquad\square$

Theorem 6 implies theorem 1 with $m = n$:

Corollary 2. *The number of facets of the polytope* $\mathrm{KR}(X)$ *(or, equivalently, the number of vertices of* $\mathrm{LIP}(X)$*) equals* $\binom{2n}{n}$.

Proof. According to Lemma 1, facets of $\mathrm{KR}(X)$ correspond to admissible trees. By Theorem 6 admissible trees are in bijective correspondence with sequences $(p_1, \ldots, p_{n+1})$ of non-negative integers which sum up to n. For any such sequence we may consider an increasing sequence $(p_1 + 1, p_1 + p_2 + 2, \ldots, p_1 + \cdots + p_n + n)$ of numbers from 1 to $2n$, thus there are exactly $\binom{2n}{n}$ of them. $\qquad\square$

4 Rearrangements

The case $m = n$ of Theorem 1 follows from Theorem 6. Analogously, the general case follows from the following

Theorem 7. *Let* $X = \{v_1, \ldots, v_{n+1}\}$, $p_1, \ldots, p_{n+1}$ *be non-negative integers which sum up to* $m \leqslant n$. *There exist exactly* $\binom{n}{m}$ *admissible graphs such that outdegree of* v_i *equals* p_i *for all* $i = 1, 2, \ldots, n + 1$.

Proof. First of all, we prove Theorem 7 in a special case.
    Let $X = \{1, \ldots, n + 1\}$. Consider the metric

$$\rho(i, j) = 1 + i/j, \; 1 \leqslant i < j \leqslant n + 1.$$

For disjoint sequences $\{x_1 < \cdots < x_k\} \subset X$, $\{y_1 < \cdots < y_k\} \subset X$ the minimum in the right hand side of (3) is attained on the increasing permutation and only on it (this is known as "rearrangement inequality"). Thus the metric ρ is generic and the graph is admissible if and only if it does not contain edges $(x_1, y_1), (x_2, y_2)$ for which $x_1 < x_2$, $y_1 > y_2$. Assume that there are exactly k positive numbers among $p_1, \ldots, p_{n+1}$, denote them $r_1, \ldots, r_k$ in the order of corresponding points on the real line. Denote by Q, $|Q| = n + 1 - k$, the set of other points of X. The statement of Theorem 7 in the case under consideration reduces to the following:
    the number of ways to choose subsets $A_1, \ldots, A_k$ in Q so that $|A_i| = r_i$ for $i = 1, \ldots, k$ and $\max(A_i) \leqslant \min(A_{i+1})$ for $i = 1, \ldots, k - 1$ equals $\binom{n}{m} = \binom{n}{\sum r_i}$.
    This is clear. Indeed, without loss of generality we may suppose that $Q = \{1, 2, \ldots, n - k + 1\}$. Then the translates $A_1, A_2 + 1, A_3 + 2, \ldots, A_k + (k - 1)$ of the sets $A_1, \ldots, A_k$ are disjoint and they form a set of size $\sum r_i = m$ in $\{1, \ldots, n\}$.

9

Now we start to deform our metric and control that the number of admissible graphs with given outdegrees does not change.

Consider the metrics on X as point of the phase space (of dimension $n(n+1)/2$:

$$PS = \{f\colon X \times X \to \mathbb{R},\ f(x,y) = f(y,x),\ f(x,x) = 0 \text{ for } x,y \in X\}\,.$$

For any sequence $x_1, y_1, \ldots, x_k, y_k$ of mutually distinct points in X we consider an *exceptional plane*

$$\sum_{i=1}^{k} f(x_i, y_i) = \sum_{i=1}^{k} f(x_i, y_{i+1}), \text{ where } y_{k+1} := y_1 \tag{4}$$

(some exceptional planes naturally coincide, we consider only one in each class).

Consider two generic metrics ρ_1, ρ_2. Theorem 4 shows that while we change a metric continuously without meeting exceptional planes, the set of admissible trees does not change. Theorem 5 guarantees that the metric remains generic.

Replace each of the metrics ρ_1, ρ_2 with ones sufficiently close to them and draw a segment between two new metrics. Almost surely (in any reasonable sense, for example, with respect to Lebesgue measure) this segment is not contained in any exceptional plane, and it does not contain points which lie in at least two exceptional planes. Thus we may suppose that when we move on this segment, we meet at most one exceptional plane simultaneously. It remains to prove that the number of admissible graphs from the statement of Theorem (7) does not change after we intersect an exceptional plane. Consider the moment of the intersection of the plane (4). Assume that before intersection the left hand side of (4) was less than the right hand side, and vice versa after intersection. Let us describe the rearrangement of the family of admissible graphs. Admissible graphs which did not contain all k edges (x_i, y_i) remain admissible (by property (iv) of Theorem 3). Graph G, which contains all these edges, is no longer admissible. It corresponds to the following graph G', which was not admissible, but became admissible: *for each index i such that G did not contain the edge (x_i, y_{i+1}), add it and remove (x_i, y_i).* Note that outdegrees do not change after such rearrangement. Let us prove that this graph G' is admissible. The old graph G was contained in some admissible tree. It contained all but one edges of the cycle $\gamma = y_1 x_1 \ldots y_k x_k y_1$, else it would remain admissible by Theorem 4. This tree is changed by replacing one edge to another, and new tree T' contains G'. Thus it suffices to check that T' is admissible. Denote by ρ the metric in the moment of rearrangement. There exists a function f with Lipschitz constant 1 such that $f(x) - f(y) = \rho(x,y)$ for all edges (x,y) of the tree T (condition (ii) of Theorem 3 and passing to the limit). Thus the same equation holds for the only edge of $T' \setminus T$ (i.e., for the edge of the cycle γ which is absent in T: this follows from the equation of the intersected plane). For other pairs of points we have a strict inequality $|f(x) - f(y)| < \rho(x,y)$. Thus the graph $T \cup T'$ is admissible in the moment of rearrangement. Denote by ρ' the metric after rearrangement. Consider a function f' such that $f'(x) - f'(y) = \rho'(x,y)$ for $(x,y) \in T'$. If $(x,y) \notin T \cup T'$, then inequality $f(x) - f(y) < \rho(x,y)$ was strict, therefore it still holds for f' and ρ'. For the the unique edge of $T' \setminus T$ it also holds, since we intersected the plane.

So, we see that the number of admissible graphs with given outdegrees does not decrease after we intersect the plane (the map $G \to G'$ is injective). Analogously, it does not increase. Theorems 7 and 1 are proved. $\qquad\square$

5   Estimates of the number of types

In this section we prove the estimates of Theorem 2.

In the previous section we considered exceptional planes. Consider also the planes determined by not necessary distinct points $x_1, \ldots, x_k, y_1, \ldots, y_k$ (but x's are distinct and y's are distinct). Denote by N the number of such planes. They partition the phase space PS onto several parts (not necessary open, for example, two points partition the line onto 5 parts: open interval, two open rays and two points). Two functions $f, g \in PS$ belong to the same part iff

$$\operatorname{sign} I(f) = \operatorname{sign} I(g)$$

for all linear functionals I, which determine exceptional planes. Note that if two metrics belong to the same part, then the families of admissible graphs for them coincide by condition (ii) of Theorem 3. The graphs $D(\alpha)$, where α is a facet of KR, are inclusion-maximal admissible graphs. Thus the families of facets for metrics ρ_1, ρ_2 coincide. Therefore the families of faces of lower dimension also coincide (faces of lower dimensions are intersections of facets). Thus the metrics ρ_1, ρ_2 are Lipschitz combinatorially equivalent. Therefore the number of types of combinatorial Lipschitz equivalence does not exceed the number of parts defined by N planes in the space of dimension $n(n+1)/2$.

We need some reasonable estimate for the number $f(N, d)$ of parts onto which N hyperplanes may divide d-dimensional Euclidian space. For example, $f(N, d) \leqslant (3N)^d$, that immediately follows from the inductive estimate $f(N+1, d) \leqslant f(N, d) + 2f(N, d-1)$. Of course, the explicit formulae similar to these of [11] for the number of open parts may be easily obtained, but the simple-looking estimate is more appropriate for us. Indeed, since, obviously, $N \leqslant n^{2n}$, we get the upper estimate in Theorem 2.

Now we come to the proof of the lower bound. Fix a function $f \in PS$ such that its values $f(x, y)$ for $x \neq y$ belong to the interval $(0, 1)$ and are linearly independent over $\mathbb{Q}$. Consider $2^{n(n+1)/2}$ metrics of the type $\rho(x, y) = 3 \pm f(x, y)$ for $x \neq y$ (for all choices of signs.) The claim is that at most $2^{o(n^2)}$ these metrics may be mutually Lipschitz combinatorially equivalent. It would immediately imply the lower bound for generic metrics. Fix a metric ρ_0 and estimate the number of metrics ρ equivalent to ρ_0. Consider a graph on X, with edges corresponding to different signs for ρ_0 and ρ. Assume that this graph contains all edges $(x_1, y_1), (x_2, y_2), (x_1, y_2), (x_2, y_1)$ of a certain cycle of length 4. Apply condition (iv) of Theorem 3 for the edges $(x_1, y_1), (x_2, y_2)$. We see that it holds for exactly one of the metrics ρ, ρ_0. Therefore ρ, ρ_0 are not Lipschitz combinatorially equivalent. Therefore the number of metrics ρ which are Lipschitz combinatorially equivalent to ρ_0 does not exceed the number of graphs on $n+1$ vertices without 4-cycles. Such a graph contains at most $(n+1)^{3/2}$ edges (see, for example, [3]), which may be chosen by at most $(n^2)^{(n+1)^{3/2}} = 2^{o(n^2)}$ ways, as desired.

6 Unimodular triangulations of the root polytope

Let ρ be a metric on a set X, $|X| = n+1$. Consider also the metric $\mathbf{1}(x,y) = 1, x \neq y$ on X. The vertices of the polytope $\mathrm{KR}((X,\rho))$ lie on rays, which go from the origin to the vertices of the polytope $\mathrm{Root}(X) := \mathrm{KR}((X,\mathbf{1}))$. By Theorem 3, admissible graphs for the metric $\mathbf{1}$ are exactly all bipartite graphs (with edges oriented from one part to another). Therefore, if the metric ρ is strict, then the graph admissible for ρ is admissible also for $\mathbf{1}$. If ρ is also generic, then for any facet of $\mathrm{KR}((X,\rho))$ we get a corresponding (under central projection) simplex belonging to some facet of $\mathrm{Root}(X)$. Thus the central projection of the boundary of $KR((X,\rho))$ onto the boundary of $\mathrm{Root}(X)$ gives a triangulation of this last simplicial complex. Consider convex hulls of the simplices of this triangulation and the origin. We get a triangulation of the polytope $\mathrm{Root}(X)$ itself. Note two properties of these triangulations. Firstly, they are *regular*, in the sense that simplices of the triangulations are the linearity set for the convex function: Kantorovich – Rubinstein norm corresponding to the metric ρ. Secondly, they are *unimodular*: all simplices in such a triangulation have equal volume. Indeed, any difference $\delta_x - \delta_y$ is expressed via analogous differences for any tree as a linear combination with coefficients ± 1; thus the linear maps which map simplices of our triangulation to each other have integer coefficients, and their determinants are equal to ± 1. It is known that all unimodular triangulations of a lattice polytope have the same f-vector (which may be defined invariantly via Ehrhart polynomial, see [4]). In turn, f-vectors of unimodular triangulations of the root polytope were calculated in [5] (for concrete triangulation, as in the present paper), and this gives another proof of Theorem 1. However we retain a combinatorial proof, which says more (Theorem 7).

From the other point of view, we may consider regular triangulations of the polytope $\mathrm{Root}(X)$, which correspond to generic metrics, and estimate the number of such triangulations from below as in section 5. Namely, fix a partition $X = X_+ \sqcup X_-$, $|X_+| = k$, $|X_-| = n+1-k$. It corresponds to a bipartite oriented graph (edge go from X_+ to X_-). In turn, it corresponds to a facet α_0 of the polytope $\mathrm{Root}(X)$, which is a product of simplices $\Delta^{k-1} \times \Delta^{n-k}$ (see, for example, 6.2.2 in [10]).

Now we proceed as in Section 5. Fix a function f on $X \times X$ such that its values $f(x,y)$ for $x \neq y$ belong to $(0,1)$ and are linearly independent over $\mathbb{Q}$. Again consider all metrics of the type $\rho(x,y) = 3 \pm f(x,y)$ for $x \neq y$ with independent choices of sign for each pair. For any such metric ρ we get a polytope $\mathrm{KR}((X,\rho))$, for this polytope we get a regular triangulation of the polytope $\mathrm{Root}(X)$ and in particular of the facet α_0. It is clear that this triangulation of α_o depends only on the choice of signs for pairs (x,y), $x \in X_+$, $y \in X_-$. Provide an estimate for the number of choices of signs such that we get a metric equivalent to a given metric ρ_0. Consider the bipartite graph with parts (X_+, X_-), in which the edges correspond to different signs chosen for ρ_0 and ρ. Note that if this graph contains a 4-cycle, the pair of opposite edges of this cycles is admissible for exactly one of two metrics ρ, ρ_0 (by condition (iv) of Theorem 3). Therefore the metrics ρ, ρ_0 define different triangulations of the facet α_0. Therefore the number of metrics ρ does not exceed the number of 4-cycle-free spanning subgraphs of the complete bipartite graph on (X_+, X_-). The number of edges on such a graph does not exceed $O(k(n-k+1)/\sqrt{n})$ (it follows from the standard argument

that any pair of vertices in one part has at most one common neighbor in another.) So we have proved

Theorem 8. *Binary logarithm of the number of regular triangulations of the product of simplices $\Delta^{k-1} \times \Delta^{n-k}$ is at least*

$$k(n - k + 1) - O(k(n - k + 1) \cdot \log(n) \cdot n^{-1/2}).$$

For small k this bound is worse than the known bounds [6].

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References

[1] A. M. Vershik. Classification of finite metric spaces and combinatorics of convex polytopes. *Arnold Mathematical Journal* **1** (1) (2015), 75–81.

[2] L. V. Kantorovich and G. S. Rubinshtein, On a space of completely additive functions. *Vestnik Leningrad. Univ.* **13** (7) (1958), 52–59 (Russian, with English summary).

[3] I. Reiman. Über ein Problem von K. Zarankiewicz. *Acta Math. Acad. Sci. Hungar.* **9** (1958), 269-278.

[4] R. Stanley. Decompositions of rational convex polytopes. *Ann. Discr. Math.* **6** (1980), 333–342.

[5] F. Ardila, M. Beck, S. Hoşten, J. Pfeifle, K. Seashore. Root Polytopes and Growth Series of Root Lattices. *SIAM J. Discrete Math.* **25** (1), 360–378.

[6] F. Santos. The Cayley trick and triangulations of products of simplices. In: Integer Points in Polyhedra. Geometry, Number Theory, Algebra, Optimization. AMS, Contemporary Mathematics 374; 2005; 191 pp.

[7] J. Melleray, F. V. Petrov and A. M. Vershik. Linearly rigid metric spaces and the embedding problem, *Fund. Math.*, **199** (2) (2008), 177–194 .

[8] P. B. Zatitskiy. On the coincidence of the canonical embeddings of a metric space into a Banach space. *J. of Math. Sci.*, **158** (6) (2009), 853–857.

[9] I. M. Gelfand, M. I. Graev and A. Postnikov. Combinatorics of hypergeometric functions associated with positive roots, in Arnold-Gelfand Mathematical Seminars, Geometry and Singularity Theory, (Eds. V. I. Arnold, I. M. Gelfand, M. Smirnov and V. S. Retakh), Birkhauser, Boston, 1997, 205–221.

[10] J. A. De Loera, J. Rambau and F. Santos. Triangulations: structures for algorithms and applications. Springer, Berlin (2010).

[11] L. Schläfli. Theorie der vielfachen Kontinuität. Denkschriften der Schweizerischen Akademie der Naturwisschenschaften, vol. 38, Birkhauser Basel, 1901.

[12] M. Develin, B. Sturmfels. Tropical convexity. *Doc. Math.*, **9**, 1–27 (2004).

[13] Th. Lam, A. Postnikov. Alcoved polytopes. I, *Discrete Comput. Geom.* **38** (3) (2007), 453–478.

[14] M. Joswig, K. Kulas. Tropical and ordinary convexity combined. *Advances in Geometry* **10** (2) (2010), 333–352.

[15] Ngoc Mai Tran. Enumerating polytropes. arXiv:1310.2012