Constant terms of near-Dyson polynomials

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Abstract

We formulate and prove a formula for the constant term for a certain class of Laurent polynomials, which include the Dyson conjecture and its generalizations by Bressoud and Goulden. Our method is explicit Combinatorial Nullstellensatz, which was used recently for finding other coefficients in [Karasev and Petrov, Károlyi and Nagy, KNPV, Ekhad and Zeilberger].

We recall the following form of Alon's Combinatorial Nullstellensatz (appeared recently in [Schauz, Lasoń, Karasev and Petrov], but essentially going back to Jacobi [Jacobi], see modern exposition in [Kunz and Kreuzer]) which proved to be very useful [Karasev and Petrov, Károlyi and Nagy, KNPV, Ekhad and Zeilberger] for finding coefficients of polynomials.

Theorem 1 (Combinatorial Nullstellensatz)**.** *Let $f(x_1, \dots, x_n)$ be a polynomial of degree $\le d_1 + \dots + d_n$.*

Consider the grid $A = \{(a_1, \dots, a_n) \mid a_i \in A_i\}$, $\#A_i = d_i + 1$. The coefficient of $\prod_{i=1}^n x_i^{d_i}$ in f is

$$\sum_{(a_1,\dots,a_n)\in A} \frac{f(a_1, \dots, a_n)}{\prod_{i=1}^n \prod_{y\in A_i\setminus a_i}(a_i - y)}.$$

Let $x_1, \dots, x_n, q$ be commuting indeterminates. Denote the coefficient of $x_1^0 \dots x_n^0$ of Laurent polynomial $f(x_1, \dots, x_n, q)$ as $CT[f]$.

Let $a_1, \dots, a_n$ be non-negative integers, $a = a_1 + \dots + a_n$. In a seminal 1962 work [Dyson] the following conjecture was put:

$$CT \prod_{i\neq j}(1 - x_i/x_j)^{a_i} = \frac{a!}{\prod a_i!} \tag{1}$$

This was proved by Gunson [unpublished] and [Wilson] in the same year. The elegant proof, based on Lagrange interpolation, is given in [Good]. In [Karasev and Petrov] another proof based on above-stated form of Combinatorial Nullstellensatz is given. It was generalized to a q-version (proved for the first time in [Zeilberger and Bressoud] by a different method) in [Károlyi and Nagy].

Constant term identities with Laurent polynomials (such as this one) often arise in quantum electrodynamics. They are also closely related to Selberg-type integrals, which play an important role in random matrix theory, statistical mechanics and special function theory (see the exposition in [Forrester and Warnaar]).

There are versions of (a particular case of) Dyson's conjecture for arbitrary root systems, in which Dyson's original case corresponds to A_n. These are famous Macdonald's conjectures proved by [I. Cherednik] with the help of the so called double affine Hecke algebras.

Understanding, for which Laurent polynomials such identities do exist, is an important question. The application of Combinatorial Nullstellensatz allowed to make substantial progress in this area, and our results continue this development.

We start from reminding the proof of q-version of Dyson conjecture.

Define $[l, r] = \{l, l+1, \dots, r\}$.

Let $\chi(\dots)$ be equal to 1 if the expression in parentheses is true, and to 0 otherwise.

Also, denote $(x)_n = \prod_{t=0}^{n-1}(1 - q^t x)$.

Theorem 2. *Let $a_1, \dots, a_n$ be non-negative integers, $a = a_1 + \dots + a_n$. Consider Laurent polynomial*

$$f(x_1, \dots, x_n, q) = \prod_{1 \le i < j \le n} (x_i/x_j)_{a_i} (q x_j/x_i)_{a_j}.$$

Then

$$CT[f] = \frac{(q)_a}{(q)_{a_1}(q)_{a_2} \cdots (q)_{a_n}}.$$

Proof. We can assume all $a_i > 0$ (if $a_i = 0$ then each factor of f contains x_i only in non-negative degree. Since we are interested in constant term of f, we can assume f does not depend on x_i).

$CT[f]$ equals to the coefficient of $\prod_{i=1}^n x_i^{a-a_i}$ of polynomial g, where

$$g(x_1, \dots, x_n, q) = \prod_{1 \le i < j \le n} (x_i/x_j)_{a_i} (q x_j/x_i)_{a_j} \times x_j^{a_i} x_i^{a_j}.$$

We will calculate this coefficient of g using Combinatorial Nullstellensatz.

Consider grid

$$R = \{(q^{y_1}, \dots, q^{y_n}) \mid 0 \le y_i \le a - a_i\}.$$

Let us assume $x = (x_1, \dots, x_n) = (q^{y_1}, \dots, q^{y_n}) = q^y \in R$ is not a zero of g. Then for each $i < j$

$$y_j - y_i \ge a_i \text{ or } y_i - y_j \ge a_j + 1,$$

otherwise one of the factors in $(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j} \times x_j^{a_i} x_i^{a_j}$ equals to zero.

In particular, it means all y_i are pairwise distinct. Let $\pi \in S_n$ be such a permutation that

$$y_{\pi_1} < y_{\pi_2} < \dots < y_{\pi_n}.$$

We know that

$$y_{\pi_{i+1}} - y_{\pi_i} \ge a_{\pi_i} + \chi(\pi_{i+1} > \pi_i).$$

Adding up these inequalities and taking into account that $y_{\pi_1} \ge 0$, we get

$$y_{\pi_n} - y_{\pi_1} \ge a - a_{\pi_n} + \sum_{i=1}^{n-1} \chi(\pi_{i+1} > \pi_i).$$

But $y_{\pi_n} \le a - a_{\pi_n}$, so $\pi_i > \pi_{i+1}$ for all i, which means $\pi = id$.

Let us note that all intermediate inequalities have to become equalities, so the only point on grid which is not a zero of g is q^y, where $y_i = a_1 + a_2 + \dots + a_{i-1}$.

Define for convenience $y_{n+1} = a$. By Combinatorial Nullstellensatz

$$CT[f] = \left(\prod_{1 \le i < j \le n} (q^{y_i - y_j})_{a_i} (q^{y_j + 1 - y_i})_{a_j} \times q^{y_j a_i} q^{y_i a_j} \right) / \left(\prod_{i=1}^n \prod_{z \in [0, a-a_i] \setminus y_i} (q^{y_i} - q^z) \right) =$$

$$= \left(\prod_{1 \le i < j \le n} \left(\prod_{k=0}^{a_i - 1} (q^{y_j} - q^{y_i + k}) \times \prod_{k=0}^{a_j - 1} (q^{y_i} - q^{y_j + 1 + k}) \right) \right) /$$

$$/\left(\prod_{i=1}^{n}(-1)^{y_i}\left(\prod_{t=0}^{y_i-1}q^t\right)(q)_{y_i}q^{y_i(a-a_i-y_i)}(q)_{a-a_i-y_i}\right)=$$

$$=\left(\prod_{1\le i<j\le n}\left((-1)^{a_i}\left(\prod_{t=y_i}^{y_{i+1}-1}q^t\right)\frac{(q)_{y_j-y_i}}{(q)_{y_j-y_{i+1}}}\times q^{y_ia_j}\frac{(q)_{y_{j+1}-y_i}}{(q)_{y_j-y_i}}\right)\right)/$$

$$/\left(\prod_{i=1}^{n}(-1)^{y_i}\left(\prod_{t=0}^{y_i-1}q^t\right)(q)_{y_i-y_1}q^{y_i(a-a_i-y_i)}(q)_{y_{n+1}-y_{i+1}}\right)=$$

$$=(q)_{y_{n+1}-y_1}\ /\left(\prod_{i=1}^{n}(q)_{y_{i+1}-y_i}\right)=\frac{(q)_a}{(q)_{a_1}\cdots(q)_{a_n}}.$$

□

Next we give simple proofs of the master theorem and its transitive analogue from [Bressoud and Goulden] using the similar technique.

A tournament T on n vertices is a set of ordered pairs (i,j) such that $1\le i\ne j\le n$ and $(i,j)\in T$ if and only if $(j,i)\notin T$. One way of interpreting a tournament is as a relation on a set $[1,n]$: $i\to j$ if and only if $(i,j)\in T$.

A tournament T is transitive if relation $\to$ is transitive. For transitive tournament T a winner permutation $\sigma\in S_n$ is such a permutation that $\sigma_i\to\sigma_j$ if and only if $i<j$.

Theorem 3. *Let T be a tournament on n vertices. Let $a_1,\dots,a_n$ be positive integers, $a=a_1+\cdots+a_n$. Consider Laurent polynomial*

$$f(x_1,\dots,x_n,q)=\prod_{(i,j)\in T}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j-1}.$$

Then $CT[f]=0$ if T is nontransitive. If T is a transitive tournament with winner permutation σ,

$$CT[f]=\frac{(q)_a}{(q)_{a_1}(q)_{a_2}\cdots(q)_{a_n}}\times\prod_{i=1}^{n}\frac{1-q^{a_{\sigma_i}}}{1-q^{a_{\sigma_1}+\cdots+a_{\sigma_i}}}.$$

Proof. Let $deg(i)=\#\{j\mid(i,j)\in T\}$. Consider a permutation $\delta\in S_n$ such that for each $1\le i<j\le n$ $deg(\delta_i)\ge deg(\delta_j)$ and $deg(\delta_i)=deg(\delta_j)$ only when $\delta_i<\delta_j$. Note that $\sigma=\delta$ in case of transitive T.

$CT[f]$ equals to the coefficient of $\prod_{i=1}^n x_i^{a-a_i-deg(i)}$ of polynomial g, where

$$g(x_1,\ldots,x_n,q)=\prod_{(i,j)\in T}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j-1}\times x_j^{a_i}x_i^{a_j-1}.$$

Once again, we will calculate this coefficient using Combinatorial Nullstellensatz.

Consider grid

$$R=\{(q^{y_1},\ldots,q^{y_n})\mid y_i\in R_i\},$$

where

$$R_i=[0,a-a_i]\setminus S_i,$$

$$S_{\delta_i}=\{a-a_{\delta_i}-\sum_{v=j}^n a_{\delta_v}\mid n+1-deg(\delta_i)<j\le n+1\}.$$

Assume $x=(x_1,\ldots,x_n)=(q^{y_1},\ldots,q^{y_n})=q^y\in R$ is not a zero of g. For each $(i,j)\in T$

$$y_j-y_i\ge a_i \text{ or } y_i-y_j\ge a_j,$$

otherwise one of the factors in $(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j-1}\times x_j^{a_i}x_i^{a_j-1}$ equals to zero.

It follows all y_i are pairwise distinct. Let $\pi\in S_n$ be such a permutation that

$$y_{\pi_1}<y_{\pi_2}<\cdots<y_{\pi_n}.$$

We know that

$$y_{\pi_{i+1}}-y_{\pi_i}\ge a_{\pi_i}.$$

Adding up these inequalities and taking into account that $y_{\pi_1}\ge 0$, we get

$$y_{\pi_n}-y_{\pi_1}\ge a-a_{\pi_n}.$$

But $y_{\pi_n}\le a-a_{\pi_n}$, so all intermediate inequalities have to become equalities and

$$y_{\pi_i}=a-\sum_{j=i}^n a_{\pi_j}.$$

Since $y_{\pi_n}\notin S_{\pi_n}$, from definition of S_{π_n} it follows $deg(\pi_n)=0$. But T is a tournament, so $deg(i)=0$ for at most one i. Since q^y is not a zero of g, such i exists (and equals to δ_n), so $deg(\delta_n)=0$ and $\pi_n=\delta_n$.

Assume we already showed that $\pi_k=\delta_k$ and $deg(\delta_k)=n-k$ for $j<k\le n$. Note that these conditions on deg imply that $(\delta_i,\delta_k)\in T$ for all

$1 \le i < k$, $j < k \le n$. Then $deg(\delta_i) \ge n - j$ for all $1 \le i \le j$, and since T is a tournament, $deg(\delta_i) > n - j$ for all $1 \le i < j$.

$y_{\pi_j} \notin S_{\pi_j}$, so $deg(\pi_j) \le n - j$. The only case in which it is possible is when $\pi_j = \delta_j$ and $deg(\delta_j) = n - j$.

Finally, either all points of R are zeros of g and $CT[f] = 0$ or $\pi = \delta$ and $deg(\delta_i) = n - i$ for all i. If the latter is the case, obviously T is transitive and $\pi = \delta = \sigma$.

The only thing left is to calculate the coefficient in case of transitive T. We will omit the calculations here since they are given in more general case in the next theorem. □

The main result is the following theorem.

Theorem 4. *Let* $1 \le l_1 \le m_1 \le r_1 < l_2 \le \cdots \le r_{k-1} < l_k \le m_k \le r_k \le n$,

$$B_i \subset C_i = \bigcup_{j=m_i+1}^{r_i} [l_i, j-2] \times j,$$

$$U_i = \left(B_i \cup ([l_i, m_i - 1] \times m_i) \cup \bigcup_{j=m_i}^{r_i - 1} (j, j+1) \right), U = \bigcup_{i=1}^{k} U_i.$$

Let $a_1, \dots, a_n$ *be positive integers,* $a = a_1 + \cdots + a_n$. *Consider Laurent polynomial*

$$f(x_1, \dots, x_n, q) = \prod_{1 \le i < j \le n} (x_i/x_j)_{a_i} (qx_j/x_i)_{a_j - \chi((i,j) \in U)}.$$

Then

$$CT[f] = \frac{(q)_a}{(q)_{a_1}(q)_{a_2} \cdots (q)_{a_n}} \times \prod_{i=1}^{k} \prod_{j=m_i}^{r_i} \frac{1 - q^{a_j}}{1 - q^{a_{l_i} + \cdots + a_j}}.$$

Remark 1. *The statement of the theorem is long and cumbersome, therefore we provide an illustration that can help to understand the idea behind the formal definitions.*

Consider correspondence between a Laurent polynomial

$$f(x_1, \dots, x_n, q) = \prod_{1 \le i < j \le n} (x_i/x_j)_{a_{i,j}} (qx_j/x_i)_{a_{j,i}}$$

and a square matrix of non-negative integers with zeroes on the main diagonal $A = \{a_{i,j}\}_{1 \le i,j \le n}$.

The polynomial from q-Dyson theorem corresponds to a matrix

$$\begin{pmatrix} 0 & a_1 & a_1 & \dots & a_1 \\ a_2 & 0 & a_2 & \dots & a_2 \\ a_3 & a_3 & 0 & \dots & a_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_n & \dots & a_n & 0 \end{pmatrix}.$$

The polynomial from transitive part of tournament theorem (for winner permutation $\sigma = id$*) corresponds to a matrix*

$$\begin{pmatrix} 0 & a_1 & a_1 & \dots & a_1 \\ a_2 - 1 & 0 & a_2 & \dots & a_2 \\ a_3 - 1 & a_3 - 1 & 0 & \dots & a_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n - 1 & a_n - 1 & \dots & a_n - 1 & 0 \end{pmatrix}.$$

The polynomial from the main theorem in case $k = 1$*,* $l_1 = 2$*,* $m_1 = 5$*,* $r_1 = 8$*,* $B_1 = \{(2,6),(4,6),(3,7),(4,7),(2,8),(4,8)\}$ *corresponds to a matrix*

$$\begin{pmatrix} 0 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & 0 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 \\ a_3 & a_3 & 0 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 & 0 & a_4 & a_4 & a_4 & a_4 & a_4 \\ a_5 & a_5 - 1 & a_5 - 1 & a_5 - 1 & 0 & a_5 & a_5 & a_5 & a_5 \\ a_6 & a_6 - 1 & a_6 & a_6 - 1 & a_6 - 1 & 0 & a_6 & a_6 & a_6 \\ a_7 & a_7 & a_7 - 1 & a_7 - 1 & a_7 & a_7 - 1 & 0 & a_7 & a_7 \\ a_8 & a_8 - 1 & a_8 & a_8 - 1 & a_8 & a_8 & a_8 - 1 & 0 & a_8 \\ a_9 & a_9 & a_9 & a_9 & a_9 & a_9 & a_9 & a_9 & 0 \end{pmatrix}.$$

As we can see, this matrix is a deformed version of q-Dyson matrix, with some coefficients decreased. The decreased coefficients are grouped into k blocks. Each block consists of the segment of some row ($[l_i, m_i - 1] \times m_i$*, marked red in the example matrix above), the segment of the diagonal (the one under the main diagonal,* $\bigcup_{j=m_i}^{r_i-1} (j, j+1)$*, also marked red) and an arbitrary subset of elements "under" them (*B_i *is an arbitrary subset of* C_i*, which is marked blue in the example).*

Remark 2. *This theorem gives classic q-Dyson theorem when $k=0$. It also gives transitive part of tournament theorem (for winner permutation $\sigma = id$) when $k=1$, $l_1 = m_1 = 1$, $r_1 = n$, $B_1 = C_1$.*

Remark 3. *This is a generalization of theorem 2.5 from [Bressoud and Goulden]. Specifically, it gives the said theorem when $B_i = C_i$ for all i.*

Proof. $CT[f]$ equals to the coefficient of

$$\prod_{i=1}^{n} x_i^{a-a_i-\sum_{j=1}^{n}\chi((i,j)\in U)}$$

of polynomial g, where

$$g(x_1,\ldots,x_n,q) = \prod_{1\le i<j\le n} (x_i/x_j)_{a_i}(qx_j/x_i)_{a_j-\chi((i,j)\in U)} \times x_j^{a_i} x_i^{a_j-\chi((i,j)\in U)}.$$

We will use Combinatorial Nullstellensatz again. Consider grid

$$R = \{(q^{y_1},\ldots,q^{y_n}) \mid y_i \in R_i\},$$

where

$$R_i = [0, a-a_i] \setminus S_i,$$

$$S_i = \{a - a_i - \sum_{v=r_t+1}^{n} a_v\} \cup \{a - a_i - \sum_{v=j}^{n} a_v \mid (i,j) \in B_t\}, \text{ if } i \in [l_t, r_t - 1] \ \exists\, t,$$

$$S_i = \emptyset \text{ otherwise.}$$

Denote $A \rtimes B = \{(i,j) \in A \times B \mid i < j\}$.
Consider

$$N = [1,n] \rtimes [1,n], V_i = [l_i, r_i] \rtimes [m_i, r_i], V = \bigcup_{i=1}^{k} V_i.$$

We replace linear factors of g

$$(x_i - q^{a_j} x_j), \text{ where } (i,j) \in V \setminus U$$

with

$$(x_i - q^{a_j} x_j - (q^{a-a_i-\sum_{v=j}^{n} a_v} - q^{a-\sum_{v=j}^{n} a_v}))$$

and call the modified polynom g'. The coefficient of g we are interested in coincides with the corresponding coefficient of g' because it has the maximal sum of degrees of x_i and the polynoms differ only by constants in linear factors.

Let

$$\chi_1(i,j) = \chi\left((i,j) \in V \setminus U \text{ and } y_i = a - a_i - \sum_{v=j}^{n} a_v \text{ and } y_j = a - \sum_{v=j}^{n} a_v\right).$$

Assume g' does not vanish at $x = q^y \in R$, then for each $i < j$ either $y_j - y_i \geq a_i + \chi_1(i,j)$ or $y_i - y_j \geq a_j + \chi((i,j) \in N \setminus V)$ (otherwise one of the linear factors of g' is zero).

It follows that all y_i are pairwise distinct. Let $\pi \in S_n$ be such a permutation that

$$y_{\pi_1} < y_{\pi_2} < \cdots < y_{\pi_n}.$$

We know that

$$y_{\pi_{i+1}} - y_{\pi_i} \geq a_{\pi_i} + \chi((\pi_{i+1}, \pi_i) \in N \setminus V) + \chi_1(\pi_i, \pi_{i+1}).$$

Adding up these inequalities and taking into account that $y_{\pi_1} \geq 0$, we get

$$y_{\pi_n} \geq a - a_{\pi_n} + \sum_{i=1}^{n-1} \chi((\pi_{i+1}, \pi_i) \in N \setminus V) + \chi_1(\pi_i, \pi_{i+1}).$$

But $y_{\pi_n} \leq a - a_{\pi_n}$, so $(\pi_{i+1}, \pi_i) \notin N \setminus V$ and $\chi_1(\pi_i, \pi_{i+1}) = 0$ for all i.

Note that all intermediate inequalities have to become equalities, so

$$y_{\pi_i} = a - \sum_{j=i}^{n} a_{\pi_j}.$$

Let us denote the event $\pi_{i+1} < \pi_i$ as descent. The descent is possible only then $(\pi_{i+1}, \pi_i) \in V$. From definition of V it follows that descents happen only if $\pi_i, \pi_{i+1} \in [l_t, r_t]$ for some t. Then for each t all elements of π in range $[l_t, r_t]$ should go in a row, all elements less than them should go before them, and all elements bigger should go after.

We will show that elements from $[l_t, r_t]$ go not just in a row but in ascending order. Therefore the only possible choice for π is id and there is only one point on grid at which g' does not vanish.

If $\pi_{r_t} \in [l_t, r_t - 1]$, then

$$y_{\pi_{r_t}} = a - a_{\pi_{r_t}} - \sum_{j=r_t+1}^{n} a_{\pi_j} = a - a_{\pi_{r_t}} - \sum_{j=r_t+1}^{n} a_j,$$

which contradicts the definition of $R_{\pi_{r_t}}$. So $\pi_{r_t} = r_t$.

Consider $s \geq m_t$ and we already showed that $\pi_{s+1} = s+1, \dots, \pi_{r_t} = r_t$. Let us assume $\pi_s < s$.

$\chi_1(\pi_s, \pi_{s+1}) = \chi_1(\pi_s, s+1) = 0$. Additionally,

$$y_{\pi_s} = a - a_{\pi_s} - \sum_{j=s+1}^{n} a_j, y_{s+1} = a - \sum_{j=s+1}^{n} a_j$$

and $(\pi_s, s+1) \in V$, then from definition of χ_1 it follows that $(\pi_s, s+1) \in U$. $\pi_s < s$, so $(\pi_s, s+1) \in B_t$. But

$$y_{\pi_s} = a - a_{\pi_s} - \sum_{j=s+1}^{n} a_j,$$

which contradicts the definition of R_{π_s}. So $\pi_s = s$.

We proved that $\pi_{m_t} = m_t, \dots, \pi_{r_t} = r_t$. By definition of V no descents are possible when $\pi_i, \pi_{i+1} \in [l_t, m_t - 1]$ so all elements of $[l_t, m_t - 1]$ also go in ascending order.

So $\pi = id$, the only point of R which is not a zero of g' is $x = q^y$, where $y = (0, a_1, a_1 + a_2, \dots, a_1 + \dots + a_{n-1})$.

Let us see what changes happened to calculation of the coefficient compared to q-Dyson theorem. For convenience denote $y_{n+1} = a$.

Fix $1 \leq t \leq k$.

Firstly, elements of S_i $(l_t \leq i < r_t)$ disappeared from R_i , so the coefficient increased in

$$\left(\prod_{i=l_t}^{r_t-1} \left(q^{y_i} - q^{a-a_i-\sum_{v=r_t+1}^{n} a_v}\right)\right) \times \prod_{(i,j)\in B_t} \left(q^{y_i} - q^{a-a_i-\sum_{v=j}^{n} a_v}\right)$$

times.

Secondly, we added linear factor

$$(x_i - q^{a_j} x_j - (q^{a-a_i-\sum_{v=j}^{n} a_v} - q^{a-\sum_{s=j}^{n} a_v}))$$

for all $(i,j) \in V_t \setminus U_t = C_t \setminus B_t$, so the coefficient increased in

$$\prod_{(i,j)\in C_t\setminus B_t} \left(q^{y_i} - q^{y_{j+1}} - (q^{a-a_i-\sum_{v=j}^n a_v} - q^{a-\sum_{v=j}^n a_v}) \right) =$$

$$= \prod_{(i,j)\in C_t\setminus B_t} \left(q^{y_i} - q^{a-a_i-\sum_{v=j}^n a_v} \right)$$

times.

Thirdly, we removed linear factor $(x_i - q^{a_j} x_j)$ for all $(i,j) \in V_t$, so the coefficient decreased in

$$\prod_{(i,j)\in V_t} (q^{y_i} - q^{y_{j+1}})$$

times.

In total, the coefficient increased in

$$\prod_{i=l_t}^{r_t-1} \left(q^{y_i} - q^{a-a_i-\sum_{v=r_t+1}^n a_v} \right) \times \prod_{(i,j)\in C_t} \left(q^{y_i} - q^{a-a_i-\sum_{v=j}^n a_v} \right) /$$

$$/ \left(\prod_{(i,j)\in V_t} (q^{y_i} - q^{y_{j+1}}) \right) =$$

$$= \prod_{j=m_t}^{r_t} \prod_{i=l_t}^{j-1} \left(q^{y_i} - q^{a-a_i-\sum_{v=j+1}^n a_v} \right) / \left(\prod_{j=m_t}^{r_t} \prod_{i=l_t}^{j-1} (q^{y_i} - q^{y_{j+1}}) \right) =$$

$$= \prod_{j=m_t}^{r_t} \prod_{i=l_t}^{j-1} (1 - q^{a_{i+1}+\cdots+a_j}) / \left(\prod_{j=m_t}^{r_t} \prod_{i=l_t}^{j-1} (1 - q^{a_i+\cdots+a_j}) \right) =$$

$$= \prod_{j=m_t}^{r_t} \prod_{i=l_t+1}^{j} (1 - q^{a_i+\cdots+a_j}) / \left(\prod_{j=m_t}^{r_t} \prod_{i=l_t}^{j-1} (1 - q^{a_i+\cdots+a_j}) \right) =$$

$$= \prod_{j=m_t}^{r_t} \frac{1 - q^{a_j}}{1 - q^{a_{l_t}+\cdots+a_j}}$$

times.

It remains to multiply the results for $1 \le t \le k$. $\square$

References

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